Une introduction aux intégrateurs variationnels en mécanique

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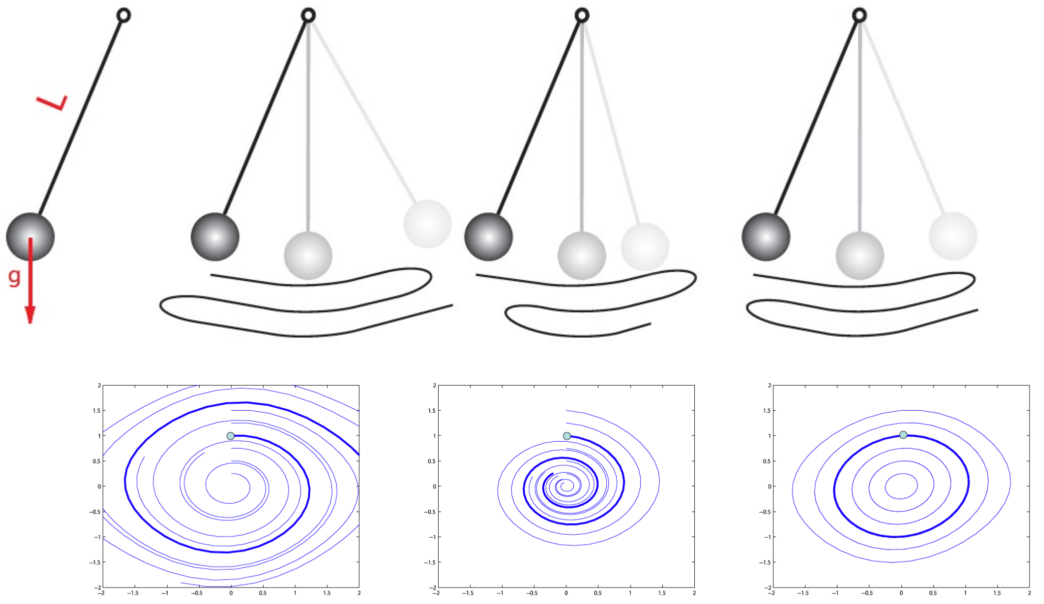
mots clés: *intégrateurs géométriques numériques, principes variationnels discrets, préservation de la structure discrète d'un système dynamique, structures symplectiques, structures multi-symplectiques, structures de Dirac, système dynamique discret, équations de conservations discrètes, symétries, mécanique de Lagrange et Hamiltonienne discrète*

1. ***Introduction aux intégrateurs temporels conventionnels***
2. ***Introduction aux intégrateurs variationnels***
3. ***Application à l'élastodynamique non-régulière***
4. ***Vers des intégrateurs variationnels pour la thermodynamique***
5. ***Cas des intégrateurs variationnels hétérogènes asynchrones***

Introduction

- *Accurate numerical simulations of dynamical systems are essential in virtually all areas of physics and mechanics.*
- *However, most traditional numerical methods do not take into account the underlying geometric structure of the physical system, leading to simulation results that may suggest non-physical behaviors.*
- *The field of geometric numerical integration (GNI) concerns numerical methods that respect the fundamental physics of a problem by preserving the geometric properties of the associated differential equations.*



Prediction is difficult, especially of the future. — Mark Twain

Introduction

- *Since the emergence of computational methods, fundamental properties such as accuracy, stability, convergence, and computational efficiency are considered crucial in deciding the usefulness of a numerical algorithm.*

  Today, aspects related to the preservation of the underlying structure are considered essential, in addition to these fundamental properties.

  One of the key ideas of approximate structure preservation is to treat the numerical method as a discrete dynamical system that approximates the flow of the continuous differential equation instead of focusing on the numerical approximation of a single trajectory .

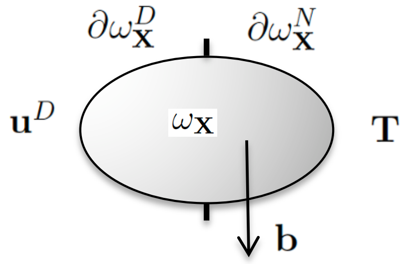
  Such an approach allows a better understanding of the invariants and the qualitative properties of the numerical method.

  MAIN GOAL: develop an explicit variational integrator for non-smooth elastodynamics satisfying exactly ALL conservation equations in the discrete sense.

Variational formalism for elastodynamics

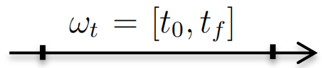
- *Deformed configuration defined by the position vector:*

$$\mathbf{x} \equiv \varphi(t, \mathbf{X}) \quad \forall (t, \mathbf{X}) \in \omega_t \times \omega_{\mathbf{X}}$$



- *State vector:*

$$\mathbf{m} \equiv (t, \mathbf{X}, \varphi)$$



- *Action of the field theory of elastodynamics (non-deformed configuration):*

$$\boxed{\mathcal{A}\left(t, \mathbf{X}, \varphi\right) = \int_{\omega_t} \int_{\omega_{\mathbf{X}}} \mathcal{L}\left(t, \mathbf{X}, \varphi, d_t\varphi, d_{\mathbf{X}}\varphi\right) \, \mathrm{d}\mathbf{X}\mathrm{d}t}$$

$d_t\varphi = \frac{d\varphi}{dt} = \mathbf{V}$ *velocity*

$d_{\mathbf{X}}\varphi = \frac{d\varphi}{d\mathbf{X}} = \mathbf{F}$ *Transformation gradient tensor*

- *Corresponding Lagrangian:*

$$\mathcal{L} = T(d_t\varphi) - V(\varphi, d_{\mathbf{X}}\varphi)$$

$$T(d_t\varphi) = \tfrac{1}{2}\rho d_t\varphi . d_t\varphi \quad \text{and} \quad V(\varphi, d_{\mathbf{X}}\varphi) = V_{int}(d_{\mathbf{X}}\varphi) - V_{ext}(\varphi)$$

$$V_{int} = \psi(d_{\mathbf{X}}\varphi) \quad \text{and} \quad V_{ext}(\varphi) = \mathbf{b}.\varphi$$

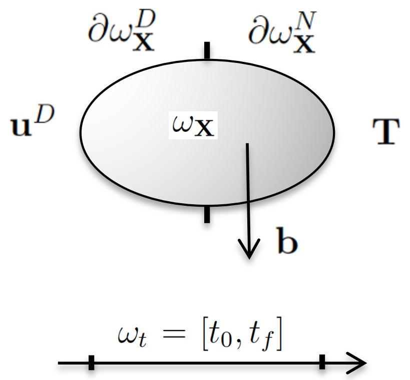
Variational formalism for elastodynamics

- *Hamilton's principle:*

$$\boxed{\begin{array}{l} \delta\mathcal{A} + \displaystyle\int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N} \mathbf{T}.\delta\varphi\,\mathrm{d}\mathbf{S}\mathrm{d}t = 0 \\ \forall\delta\mathbf{m} \in \mathcal{M}_0(\delta\mathbf{m} \in H^1(\omega_t \times \omega_{\mathbf{X}}); \delta t|_{\partial\omega_t} = 0; \delta\mathbf{X}|_{\partial\omega_{\mathbf{X}}} = 0; \ldots \\ \ldots\delta\varphi|_{\partial\omega_t} = 0; \delta\varphi|_{\partial\omega_{\mathbf{X}}^D} = 0) \end{array}}$$

$$\partial\omega_{\mathbf{X}} = \partial\omega_{\mathbf{X}}^D \cup \partial\omega_{\mathbf{X}}^N \quad and \quad \partial\omega_{\mathbf{X}}^D \cap \partial\omega_{\mathbf{X}}^N = \emptyset$$

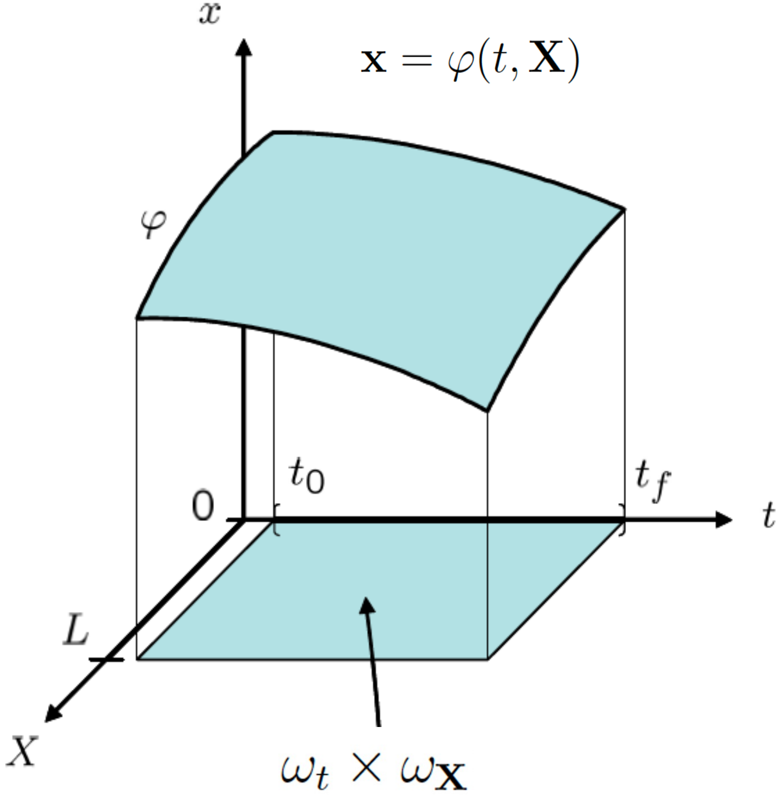
$$\varphi|_{\partial\omega_{\mathbf{X}}^D} = \mathbf{u}^D$$



- *Corresponding generators:*

$$\delta\mathcal{A} = \frac{\partial\mathcal{A}}{\partial\mathbf{m}}.\delta\mathbf{m} = \delta_t\mathcal{A} + \delta_{\mathbf{X}}\mathcal{A} + \delta_\varphi\mathcal{A} = D_t\mathcal{A}.\delta t + D_{\mathbf{X}}\mathcal{A}.\delta\mathbf{X} + D_\varphi\mathcal{A}.\delta\varphi$$

Variational formalism for elastodynamics



Variational formalism for elastodynamics

- *Vertical variation:* $\boxed{\delta\mathbf{m} \equiv (\delta t = 0, \delta\mathbf{X} = 0, \delta\varphi)}$

- *Determination of the vertical generator* $D_\varphi\mathcal{A}$ *:*

$$
\begin{aligned}
\delta_\varphi\mathcal{A} &= \int_{\omega_t}\int_{\omega_{\mathbf{X}}} \left(\frac{\partial\mathcal{L}}{\partial\varphi}.\delta\varphi + \frac{\partial\mathcal{L}}{\partial d_t\varphi}.\delta d_t\varphi + \frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi} : \delta d_{\mathbf{x}}\varphi\right) \mathrm{d}\mathbf{X}\mathrm{d}t \\
&= \int_{\omega_t}\int_{\omega_{\mathbf{X}}} \left(\frac{\partial\mathcal{L}}{\partial\varphi}.\delta\varphi\right) \mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_t}\int_{\omega_{\mathbf{X}}} d_t\left(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.\delta\varphi\right) \mathrm{d}\mathbf{X}\mathrm{d}t - \int_{\omega_t}\int_{\omega_{\mathbf{X}}} d_t\left(\frac{\partial\mathcal{L}}{\partial d_t\varphi}\right).\delta\varphi\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_t}\int_{\omega_{\mathbf{X}}} d_{\mathbf{x}}.\left(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.\delta\varphi\right) \mathrm{d}\mathbf{X}\mathrm{d}t - \int_{\omega_t}\int_{\omega_{\mathbf{X}}} d_{\mathbf{x}}.\left(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}\right).\delta\varphi\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&= \int_{\omega_t}\int_{\omega_{\mathbf{X}}} \left(\frac{\partial\mathcal{L}}{\partial\varphi}.\delta\varphi\right) \mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \left[\int_{\omega_{\mathbf{X}}} \left(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.\delta\varphi\right) \mathrm{d}\mathbf{X}\right]_{t_0}^{t_f} - \int_{\omega_t}\int_{\omega_{\mathbf{X}}} d_t\left(\frac{\partial\mathcal{L}}{\partial d_t\varphi}\right).\delta\varphi\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}} \left(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.\mathbf{N}\right).\delta\varphi\,\mathrm{d}\mathbf{S}\mathrm{d}t - \int_{\omega_t}\int_{\omega_{\mathbf{X}}} d_{\mathbf{x}}.\left(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}\right).\delta\varphi\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&= D_\varphi\mathcal{A}.\delta\varphi
\end{aligned}
$$

Variational formalism for elastodynamics

- *Vertical variation:* $\delta \mathbf{m} \equiv (\delta t = 0, \delta \mathbf{X} = 0, \delta \varphi)$

- *Associated generator:*

$$\delta_\varphi \mathcal{A} = D_\varphi \mathcal{A}.\delta\varphi$$
$$= \int_{\omega_t} \int_{\omega_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial \varphi} - d_t \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi} \right) - d_{\mathbf{x}}. \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi} \right) \right) .\delta\varphi \, \mathrm{d}\mathbf{X} \mathrm{d}t$$
$$+ \int_{\omega_t} \int_{\partial \omega_{\mathbf{X}}^N} \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi} .\mathbf{N} \right) .\delta\varphi \, \mathrm{d}\mathbf{S} \mathrm{d}t$$

- *Hamilton's principle:*

$$\boxed{D_\varphi \mathcal{A}.\delta\varphi + \int_{\omega_t} \int_{\partial \omega_{\mathbf{X}}^N} \mathbf{T}.\delta\varphi \, \mathrm{d}\mathbf{S} \mathrm{d}t = 0 \quad \forall \delta\varphi \in \mathcal{M}_0}$$

- *Euler-Lagrange equations for elastodynamics:*

$$\boxed{\begin{array}{l} \frac{\partial \mathcal{L}}{\partial \varphi} - d_t \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi} \right) - d_{\mathbf{x}}. \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi} \right) = 0 \quad in \ \omega_{\mathbf{X}} \\ \frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi} .\mathbf{N} + \mathbf{T} = 0 \quad on \ \partial \omega_{\mathbf{X}}^N \end{array}}$$

Variational formalism for elastodynamics

- *Expressions of the first derivatives of the Lagrangian:*

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \mathbf{b}$$

$$\frac{\partial \mathcal{L}}{\partial d_t \varphi} = \rho d_t \varphi = \rho \dot{\varphi} = \mathbf{p}$$ *Momentum*

$$\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi} = -\frac{\partial \psi}{\partial d_{\mathbf{x}} \varphi} = -\mathbf{\Pi}$$ *Non-symmetric stress tensor of Piola–Kirchhoff 1*

- *Field equations of elastodynamics:*

$$\boxed{\begin{array}{l} \mathbf{b} - d_t \mathbf{p} + d_{\mathbf{x}} . \mathbf{\Pi} = 0 \quad in\ \omega_t \times \omega_{\mathbf{X}} \\ -\mathbf{\Pi} . \mathbf{N} + \mathbf{T} = 0 \quad in\ \omega_t \times \partial \omega_{\mathbf{X}}^N \end{array}}$$

Variational formalism for elastodynamics

- *Horizontal variation:* $\boxed{\delta\mathbf{m} \equiv (\delta t, \delta\mathbf{X}, \delta\varphi = 0)}$

- *Determination of horizontal generators* $D_t\mathcal{A}$ *and* $D_{\mathbf{X}}\mathcal{A}$ *:*

$$
\begin{aligned}
&\delta_t\mathcal{A} + \delta_{\mathbf{X}}\mathcal{A} = \\
&\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}}{\partial t}\delta t + \frac{\partial\mathcal{L}}{\partial\varphi}.d_t\varphi\delta t + \frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_{tt}\varphi\delta t + \frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_{t\mathbf{x}}\varphi\delta t\right)\mathrm{d}\mathbf{X}\mathrm{d}t \\
&+\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{X}}.\delta\mathbf{X} + \frac{\partial\mathcal{L}}{\partial\varphi}.d_{\mathbf{x}}\varphi.\delta\mathbf{X} + \frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_{\mathbf{x}t}\varphi.\delta\mathbf{X} + \frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi} : d_{\mathbf{x}\mathbf{x}}\varphi.\delta\mathbf{X}\right)\mathrm{d}\mathbf{X}\mathrm{d}t \\
&+\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\mathcal{L}\,\delta\left(\mathrm{d}\mathbf{X}\mathrm{d}t\right) = \delta\mathcal{A}^1 + \delta\mathcal{A}^2 + \delta\mathcal{A}^3
\end{aligned}
$$

with:

$$
\begin{cases}
\delta\mathcal{A}^1 = \int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}}{\partial t} + d_t\left(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_t\varphi\right) + d_{\mathbf{x}}.\left(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_t\varphi\right)\right)\delta t \\
\quad + \left(\frac{\partial\mathcal{L}}{\partial\varphi} - d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}) - d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi})\right).d_t\varphi\delta t\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
\delta\mathcal{A}^2 = \int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{X}} + d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_{\mathbf{x}}\varphi) + d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_{\mathbf{x}}\varphi)\right).\delta\mathbf{X} \\
+ \left(\frac{\partial\mathcal{L}}{\partial\varphi} - d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}) - d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi})\right).d_{\mathbf{x}}\varphi.\delta\mathbf{X}\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
\delta\mathcal{A}^3 = -\int_{\omega_t}\int_{\omega_{\mathbf{X}}}((d_{\mathbf{X}}\mathcal{L}).\delta\mathbf{X} + (d_t\mathcal{L})\delta t)\mathrm{d}\mathbf{X}\mathrm{d}t
\end{cases}
$$

Variational formalism for elastodynamics

- *Note on the third term:*

$$\delta\mathcal{A}^3 = \int_{\omega_\tau}\int_{\omega_\Xi}\mathcal{L}\,\delta\left(d_\Xi\mathbf{X}d_\tau t\right)\mathrm{d}\Xi\mathrm{d}\tau$$

$$\forall(\delta t(\tau),\delta\mathbf{X}(\Xi))\in\square_0$$

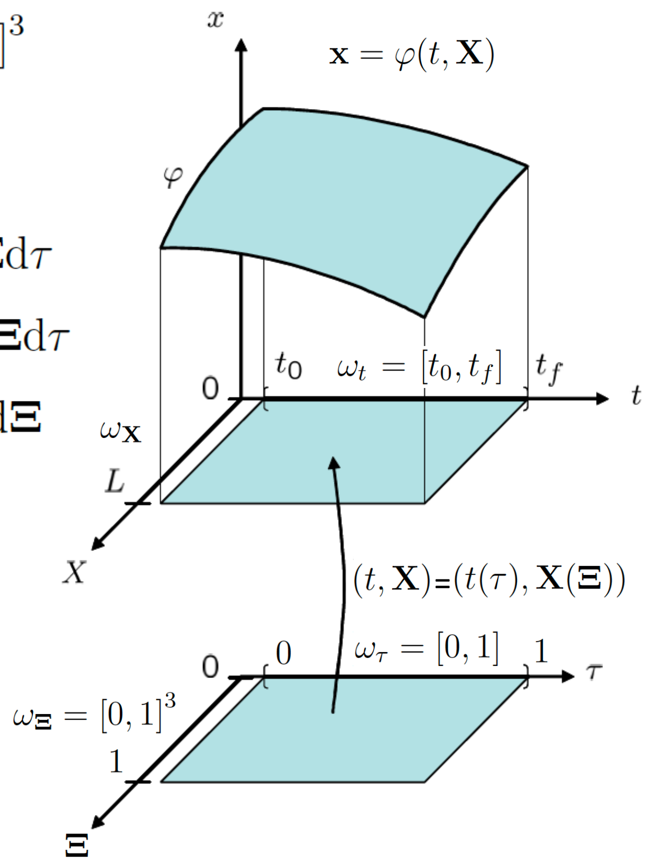
$$\square_0\equiv((\delta t(\tau),\delta\mathbf{X}(\Xi))\in H^1(\omega_\tau\times\omega_\Xi);\,\delta t(\tau)|_{\partial\omega_\tau}=0;\,\delta\mathbf{X}(\Xi)|_{\partial\omega_\Xi}=0)$$

- *Parametric configuration:* $\omega_\tau=[0,1]$ and $\omega_\Xi=[0,1]^3$

$$\delta\mathcal{A}^3=\int_{\omega_\tau}\int_{\omega_\Xi}\mathcal{L}\left(\delta(d_\Xi\mathbf{X})d_\tau t+d_\Xi\mathbf{X}\delta(d_\tau t)\right)\mathrm{d}\Xi\mathrm{d}\tau$$
$$=\int_{\omega_\tau}\int_{\omega_\Xi}\mathcal{L}(d_\Xi\delta\mathbf{X})d_\tau t\mathrm{d}\Xi\mathrm{d}\tau+\int_{\omega_\tau}\int_{\omega_\Xi}\mathcal{L}(d_\tau\delta t)d_\Xi\mathbf{X}\mathrm{d}\Xi\mathrm{d}\tau$$
$$=\int_{\omega_\tau}\int_{\omega_\Xi}d_\Xi.(\mathcal{L}\delta\mathbf{X})d_\tau t\mathrm{d}\Xi\mathrm{d}\tau-\int_{\omega_\tau}\int_{\omega_\Xi}(d_\Xi\mathcal{L}).\delta\mathbf{X}d_\tau t\mathrm{d}\Xi\mathrm{d}\tau$$
$$+\int_{\omega_\Xi}\int_{\omega_\tau}d_\tau(\mathcal{L}\delta t)d_\Xi\mathbf{X}\mathrm{d}\tau\mathrm{d}\Xi-\int_{\omega_\Xi}\int_{\omega_\tau}(d_\tau\mathcal{L})\delta t d_\Xi\mathbf{X}\mathrm{d}\tau\mathrm{d}\Xi$$

with:
$$\begin{cases}\mathbf{x}=\varphi(t,\mathbf{X})=\varphi(t(\tau),\mathbf{X}(\Xi))\\ \mathrm{d}\mathbf{X}\mathrm{d}t=(d_\Xi\mathbf{X}d_\tau t)\mathrm{d}\Xi\mathrm{d}\tau\end{cases}$$

$$\boxed{\delta\mathcal{A}^3=-\int_{\omega_t}\int_{\omega_\mathbf{X}}((d_\mathbf{X}\mathcal{L}).\delta\mathbf{X}+(d_t\mathcal{L})\delta t)\mathrm{d}\mathbf{X}\mathrm{d}t}$$



Variational formalism for elastodynamics

- *Hamilton's principle:*

$$
\begin{aligned}
&\delta_t\mathcal{A} + \delta_{\mathbf{x}}\mathcal{A} = \\
&\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}}{\partial t} + d_t\left(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_t\varphi\right) + d_{\mathbf{x}}.\left(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_t\varphi\right) - d_t\mathcal{L}\right)\delta t\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&+\int_{\omega_t}\int_{\omega_{\mathbf{X}}}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{X}} + d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d^s_{\mathbf{x}}\varphi) + d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_{\mathbf{x}}\varphi) - d_{\mathbf{X}}\mathcal{L}\right).\delta\mathbf{X}\mathrm{d}\mathbf{X}\mathrm{d}t \\
&= D_t\mathcal{A}.\delta t + D_{\mathbf{X}}\mathcal{A}.\delta\mathbf{X} = 0
\end{aligned}
$$

- *Conservation of mechanical energy (horizontal time generator):*

$$
\boxed{\frac{\partial\mathcal{L}}{\partial t} + d_t\left(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_t\varphi - \mathcal{L}\right) + d_{\mathbf{x}}.\left(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_t\varphi\right) = 0}
$$

- *Conservation of configurational forces (horizontal space generator):*

$$
\boxed{\frac{\partial\mathcal{L}}{\partial\mathbf{X}} + d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_{\mathbf{x}}\varphi - \mathcal{L}\mathbf{I}) + d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.d_{\mathbf{x}}\varphi) = 0}
$$

Variational formalism for elastodynamics

- *Space-time matrix formulation:*

$$\boxed{\left(\frac{\partial \mathcal{L}}{\partial t}, \frac{\partial \mathcal{L}}{\partial \mathbf{X}}\right) + (d_t, d_{\mathbf{x}}).\begin{bmatrix} \frac{\partial \mathcal{L}}{\partial d_t\varphi}.d_t\varphi - \mathcal{L} & \frac{\partial \mathcal{L}}{\partial d_t\varphi}.d_{\mathbf{x}}\varphi \\ \frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_t\varphi & \frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.d_{\mathbf{x}}\varphi - \mathcal{L}\mathbf{I} \end{bmatrix} = (0,0)}$$

- *Energy momentum tensor (space-time Eshelby tensor) for elastodynamics:*

$$\left(\frac{\partial \mathcal{L}}{\partial t}, \frac{\partial \mathcal{L}}{\partial \mathbf{X}}\right) + (d_t, d_{\mathbf{x}}).\mathbf{C}_{t\mathbf{x}} = (0,0)$$

- *Cases where the Lagrangian does not explicitly depend on time and space:* $\mathcal{L}\left(\varphi, d_t\varphi, d_{\mathbf{x}}\varphi\right)$

$$\boxed{\begin{aligned} & d_t(T+V) = d_{\mathbf{x}}.(\mathbf{\Pi}.\mathbf{V}) && \textit{(mechanical energy)} \\ & d_{\mathbf{x}}.(\mathbf{C}) - \mathbf{F}^T.\mathbf{b} + d_t(\mathbf{F}^T.\mathbf{p}) = 0 && \textit{(configurational forces)} \\ & \mathbf{C} = (-T+\psi)\mathbf{I} - \mathbf{\Pi}^T.\mathbf{F} && \textit{(Eshelby dynamic tensor)} \end{aligned}}$$

- *Integral form of mechanical energy conservation:*

$$\boxed{\left[\int_{\omega_{\mathbf{X}}} (T+V)\,\mathrm{d}\mathbf{X}\right]_{t_0}^{t_f} = \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^D} \mathbf{\Pi}.\mathbf{N}.\mathbf{V}\,\mathrm{d}\mathbf{S}\mathrm{d}t + \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N} \mathbf{T}.\mathbf{V}\,\mathrm{d}\mathbf{S}\mathrm{d}t}$$

Variational formalism for elastodynamics

- *Vertical generator for any subdomain:* $\omega'_{\mathbf{X}} \subset \omega_{\mathbf{X}}$

$$\delta_\varphi \mathcal{A}' = D_\varphi \mathcal{A}'.\delta\varphi = \left[\int_{\omega'_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi}\right).\delta\varphi \,\mathrm{d}\mathbf{X}\right]_{t_0}^{t_f} + \int_{\omega_t}\int_{\partial\omega'_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}.\mathbf{N}\right).\delta\varphi \,\mathrm{d}\mathbf{S}\mathrm{d}t$$
$$- \int_{\omega_t}\int_{\omega'_{\mathbf{X}}} \frac{\partial \mathcal{L}}{\partial \varphi}.\delta\varphi \,\mathrm{d}\mathbf{X}\mathrm{d}t$$

- *Noether's theorem:*

$$\boxed{\frac{d\mathcal{A}'}{d\epsilon}(\varphi^\epsilon)|_{\epsilon=0} = D_\varphi \mathcal{A}'.\frac{d\varphi^\epsilon}{d\epsilon}|_{\epsilon=0} = 0 \quad \forall \delta\varphi^\epsilon \in \mathcal{M} \equiv (\delta\varphi^\epsilon \in H^1(\omega_t \times \omega'_{\mathbf{X}}))}$$

- *For any vector* $\mathbf{v}$ *of* R^3 *:* $\boxed{\varphi^\epsilon = \varphi + \epsilon\mathbf{v}}$

$$\frac{d\mathcal{A}'}{d\epsilon}(\varphi^\epsilon)|_{\epsilon=0} = D_\varphi \mathcal{A}'.\mathbf{v}$$
$$= \left[\int_{\omega'_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial d_t \varphi}\right).\mathbf{v} \,\mathrm{d}\mathbf{X}\right]_{t_0}^{t_f} + \int_{\omega_t}\int_{\partial\omega'_{\mathbf{X}}} \left(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}.\mathbf{N}\right).\mathbf{v} \,\mathrm{d}\mathbf{S}\mathrm{d}t - \int_{\omega_t}\int_{\omega'_{\mathbf{X}}} \frac{\partial \mathcal{L}}{\partial \varphi}.\mathbf{v} \,\mathrm{d}\mathbf{X}\mathrm{d}t$$
$$= \int_{\omega_t}\int_{\omega'_{\mathbf{X}}} (d_t(\frac{\partial \mathcal{L}}{\partial d_t \varphi}.\mathbf{v}) + d_{\mathbf{x}}.(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}.\mathbf{v}) - \frac{\partial \mathcal{L}}{\partial \varphi}.\mathbf{v}) \,\mathrm{d}\mathbf{X}\mathrm{d}t$$
$$= \int_{\omega_t}\int_{\omega'_{\mathbf{X}}} (d_t(\frac{\partial \mathcal{L}}{\partial d_t \varphi}) + d_{\mathbf{x}}.(\frac{\partial \mathcal{L}}{\partial d_{\mathbf{x}} \varphi}) - \frac{\partial \mathcal{L}}{\partial \varphi}).\mathbf{v} \,\mathrm{d}\mathbf{X}\mathrm{d}t = 0 \quad \forall \mathbf{v} \in R^3$$

→ *We again obtain the conservation of the linear momentum*

Variational formalism for elastodynamics

- *For any antisymmetric tensor* $\mathbf{\Omega}$ *:* $\boxed{\varphi^\epsilon = exp(\mathbf{\Omega}\epsilon)\varphi}$

$$
\begin{aligned}
&\frac{d\mathcal{A}'}{d\epsilon}(\varphi^\epsilon)|_{\epsilon=0} = D_\varphi \mathcal{A}'.\mathbf{\Omega}\varphi \\
&= \int_{\omega_t}\int_{\omega'_{\mathbf{X}}} (d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}.\mathbf{\Omega}\varphi) + d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}.\mathbf{\Omega}\varphi) - \frac{\partial\mathcal{L}}{\partial\varphi}.\mathbf{\Omega}\varphi)\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&= \int_{\omega_t}\int_{\omega'_{\mathbf{X}}} (d_t(\frac{\partial\mathcal{L}}{\partial d_t\varphi}) + d_{\mathbf{x}}.(\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi}) - \frac{\partial\mathcal{L}}{\partial\varphi}).\mathbf{\Omega}\varphi\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_t}\int_{\omega'_{\mathbf{X}}} \frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi} : (\mathbf{\Omega} d_{\mathbf{x}}\varphi)\,\mathrm{d}\mathbf{X}\mathrm{d}t \\
&= 0 \quad \forall\mathbf{\Omega}
\end{aligned}
$$

It comes:

$$
\begin{aligned}
&\frac{\partial\mathcal{L}}{\partial d_{\mathbf{x}}\varphi} : (\mathbf{\Omega} d_{\mathbf{x}}\varphi) = 0\ \forall\mathbf{\Omega} \iff (\mathbf{\Pi}.\mathbf{F}^T) : \mathbf{\Omega} = 0\ \forall\mathbf{\Omega} \\
&\iff \mathbf{\Pi}.\mathbf{F}^T = (\mathbf{\Pi}.\mathbf{F}^T)^T
\end{aligned}
$$

$\boxed{\sigma = \sigma^T}$ *conservation of angular momentum* $\begin{cases} J = det(\mathbf{F}) \\ \sigma = J^{-1}\mathbf{\Pi}.\mathbf{F}^T \end{cases}$

➡ *Noether's theorem can also be applied to horizontal generators (e.g. rotational invariance causes Eshelby's dynamic tensor symmetry)*

$$\boxed{\mathbf{C} = (-T + \psi)\mathbf{I} - \mathbf{\Pi}^T.\mathbf{F}}$$

Multi-symplectic formalism for elastodynamics

- *Partial Legendre–Fenchel transforms (see Eshelby space-time tensor):*

$$\boxed{\begin{aligned} T^*\left(\frac{\partial T}{\partial d_t\varphi}\right) &= \sup_{d_t\varphi}\left(\frac{\partial T}{\partial d_t\varphi}.d_t\varphi - T(d_t\varphi)\right) \\ \psi^*\left(\frac{\partial \psi}{\partial d_{\mathbf{x}}\varphi}\right) &= \sup_{d_{\mathbf{x}}\varphi}\left(\frac{\partial \psi}{\partial d_{\mathbf{x}}\varphi} : d_{\mathbf{x}}\varphi - \psi(d_{\mathbf{x}}\varphi)\right) \end{aligned}}$$

$$\begin{aligned} T^*(\mathbf{p}) &= \sup_{d_t\varphi}(\mathbf{p}.d_t\varphi - T(d_t\varphi)) \\ \psi^*(\mathbf{\Pi}) &= \sup_{d_{\mathbf{x}}\varphi}(\mathbf{\Pi} : d_{\mathbf{x}}\varphi - \psi(d_{\mathbf{x}}\varphi)) \end{aligned}$$

- *Action with two partial dualizations on momentum and stress fields:*

$$\boxed{\begin{aligned} &\mathcal{A}^*(t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}) = \\ &\int_{\omega_t}\int_{\omega_{\mathbf{X}}} (\mathbf{p}.d_t\varphi - \mathbf{\Pi} : d_{\mathbf{x}}\varphi - \mathcal{L}^*(t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}))\, \mathrm{d}\mathbf{X}\mathrm{d}t \\ &\mathcal{L}^* = T^*(\mathbf{p}) - \psi^*(\mathbf{\Pi}) - V_{ext}(\varphi) \qquad T^*(\mathbf{p}) = \tfrac{1}{2}\rho^{-1}\mathbf{p}.\mathbf{p} \end{aligned}}$$

- *Corresponding state vector:* $\mathbf{m}^* \equiv (t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi})$

- *Hamilton's principle:*

$$\boxed{\begin{aligned} &\delta\mathcal{A}^* + \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N} \mathbf{T}.\delta\varphi\, \mathrm{d}\mathbf{S}\mathrm{d}t = 0 \quad \forall \delta\mathbf{m}^* \in \mathcal{M}_0^* \\ &\mathcal{M}_0^* \equiv (\delta\mathbf{m}^* \in H^1(\omega_t \times \omega_{\mathbf{X}}); \delta t|_{\partial\omega_t} = 0; \delta\mathbf{X}|_{\partial\omega_{\mathbf{X}}} = 0; \ldots \\ &\ldots\delta(\varphi, \mathbf{p}, \mathbf{\Pi})|_{\partial\omega_t} = 0; \delta\varphi|_{\partial\omega_{\mathbf{X}}^D} = 0) \end{aligned}}$$

Multi-symplectic formalism for elastodynamics

- *Notations of the 5 corresponding generators (2 horizontal and 3 vertical):*

$$\boxed{\begin{aligned}&\delta\mathcal{A}^* = \frac{\partial\mathcal{A}^*}{\partial\mathbf{m}^*}.\delta\mathbf{m}^* = \delta_t\mathcal{A}^* + \delta_\mathbf{X}\mathcal{A}^* + \delta_\varphi\mathcal{A}^* + \delta_\mathbf{p}\mathcal{A}^* + \delta_\mathbf{\Pi}\mathcal{A}^*\\ &= D_t\mathcal{A}^*.\delta t + D_\mathbf{X}\mathcal{A}^*.\delta\mathbf{X} + D_\varphi\mathcal{A}^*.\delta\varphi + D_\mathbf{p}\mathcal{A}^*.\delta\mathbf{p} + D_\mathbf{\Pi}\mathcal{A}^*.\delta\mathbf{\Pi}\end{aligned}}$$

- *Calculation of the 3 vertical generators:*

$$\begin{aligned}
&\delta_\varphi\mathcal{A}^* + \delta_\mathbf{p}\mathcal{A}^* + \delta_\mathbf{\Pi}\mathcal{A}^* =\\
&\int_{\omega_t}\int_{\omega_\mathbf{X}} (\delta\mathbf{p}.d_t\varphi + \mathbf{p}.\delta d_t\varphi - \delta\mathbf{\Pi} : d_\mathbf{x}\varphi - \mathbf{\Pi} : \delta d_\mathbf{x}\varphi)\, \mathrm{d}\mathbf{X}\mathrm{d}t\\
&- \int_{\omega_t}\int_{\omega_\mathbf{X}} \left(\frac{\partial\mathcal{L}^*}{\partial\varphi}.\delta\varphi + \frac{\partial\mathcal{L}^*}{\partial\mathbf{p}}.\delta\mathbf{p} + \frac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}}.\delta\mathbf{\Pi}\right) \mathrm{d}\mathbf{X}\mathrm{d}t\\
&\iff \int_{\omega_t}\int_{\omega_\mathbf{X}} (\delta\mathbf{p}.d_t\varphi - d_t\mathbf{p}.\delta\varphi - \delta\mathbf{\Pi} : d_\mathbf{x}\varphi + d_\mathbf{x}.\mathbf{\Pi}.\delta\varphi)\, \mathrm{d}\mathbf{X}\mathrm{d}t\\
&+ \int_{\omega_t}\int_{\partial\omega_\mathbf{X}^N} (-\mathbf{\Pi}.\mathbf{N}).\delta\varphi\, \mathrm{d}\mathbf{S}\mathrm{d}t\\
&- \int_{\omega_t}\int_{\omega_\mathbf{X}} \left(\frac{\partial\mathcal{L}^*}{\partial\varphi}.\delta\varphi + \frac{\partial\mathcal{L}^*}{\partial\mathbf{p}}.\delta\mathbf{p} + \frac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} : \delta\mathbf{\Pi}\right) \mathrm{d}\mathbf{X}\mathrm{d}t\\
&\iff \int_{\omega_t}\int_{\omega_\mathbf{X}} (-\frac{\partial\mathcal{L}^*}{\partial\varphi} - d_t\mathbf{p} + d_\mathbf{x}.\mathbf{\Pi}).\delta\varphi\, \mathrm{d}\mathbf{X}\mathrm{d}t\\
&+ \int_{\omega_t}\int_{\partial\omega_\mathbf{X}^N} (-\mathbf{\Pi}.\mathbf{N}).\delta\varphi\, \mathrm{d}\mathbf{S}\mathrm{d}t\\
&+ \int_{\omega_t}\int_{\omega_\mathbf{X}} (-\frac{\partial\mathcal{L}^*}{\partial\mathbf{p}} + d_t\varphi).\delta\mathbf{p}\, \mathrm{d}\mathbf{X}\mathrm{d}t\\
&+ \int_{\omega_t}\int_{\omega_\mathbf{X}} (-\frac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} - d_\mathbf{x}\varphi) : \delta\mathbf{\Pi}\, \mathrm{d}\mathbf{X}\mathrm{d}t
\end{aligned}$$

Multi-symplectic formalism for elastodynamics

- *Hamilton's principle:*

$$D_\varphi \mathcal{A}^*.\delta\varphi + D_{\mathbf{p}}\mathcal{A}^*.\delta\mathbf{p} + D_{\mathbf{\Pi}}\mathcal{A}^*.\delta\mathbf{\Pi} + \int_{\omega_t}\int_{\partial\omega_{\mathbf{X}}^N} \mathbf{T}.\delta\varphi \, \mathrm{d}\mathbf{S}\mathrm{d}t = 0$$

$$\forall(\delta\varphi, \delta\mathbf{p}, \delta\mathbf{\Pi}) \in \mathcal{M}_0^*$$

- *Euler-Lagrange equations of 3-fields elasto-ynamics:* $\mathbf{s} \equiv (\varphi, \mathbf{p}, \mathbf{\Pi})$

$$\boxed{\begin{array}{lll} -\frac{\partial \mathcal{L}^*}{\partial \varphi} - d_t\mathbf{p} + d_{\mathbf{x}}.\mathbf{\Pi} & = 0 & in\ \omega_t \times \omega_{\mathbf{X}} \\ -\frac{\partial \mathcal{L}^*}{\partial \mathbf{p}} + d_t\varphi & = 0 & in\ \omega_t \times \omega_{\mathbf{X}} \\ -\frac{\partial \mathcal{L}^*}{\partial \mathbf{\Pi}} - d_{\mathbf{x}}\varphi & = 0 & in\ \omega_t \times \omega_{\mathbf{X}} \\ -\mathbf{\Pi}.\mathbf{N} + \mathbf{T} & = 0 & on\ \omega_t \times \partial\omega_{\mathbf{X}}^N \end{array}}$$

- *Multi-symplectic formalism:*

$$\boxed{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} d_t \begin{bmatrix} \varphi \\ \mathbf{p} \\ \mathbf{\Pi} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} d_{\mathbf{x}} \begin{bmatrix} \varphi \\ \mathbf{p} \\ \mathbf{\Pi} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{L}^*}{\partial \varphi} \\ \frac{\partial \mathcal{L}^*}{\partial \mathbf{p}} \\ \frac{\partial \mathcal{L}^*}{\partial \mathbf{\Pi}} \end{bmatrix}}$$

$$\mathbf{W}.d_t\mathbf{s} + \mathbf{K}.d_{\mathbf{x}}\mathbf{s} = \partial_{\mathbf{s}}\mathcal{L}^*$$

Multi-symplectic formalism for elastodynamics

- *Expressions for the first derivatives of the Lagrangian:*

$$\frac{\partial \mathcal{L}^*}{\partial \varphi} = -\mathbf{b}$$
$$\frac{\partial \mathcal{L}^*}{\partial \mathbf{p}} = \frac{\partial T^*}{\partial \mathbf{p}} = \rho^{-1}\mathbf{p}$$
$$\frac{\partial \mathcal{L}^*}{\partial \mathbf{\Pi}} = -\frac{\partial \psi^*}{\partial \mathbf{\Pi}} = -\mathbf{F}$$

- *Euler-Lagrange equations of 3-field elastodynamics:*

$$\boxed{\begin{array}{lll} \mathbf{b} - d_t\mathbf{p} + d_{\mathbf{X}}.\mathbf{\Pi} & = 0 & in\ \omega_t \times \omega_{\mathbf{X}} \\ -\rho^{-1}\mathbf{p} + d_t\varphi & = 0 & in\ \omega_t \times \omega_{\mathbf{X}} \\ \mathbf{F} - d_{\mathbf{X}}\varphi & = 0 & in\ \omega_t \times \omega_{\mathbf{X}} \\ -\mathbf{\Pi}.\mathbf{N} + \mathbf{T} & = 0 & in\ \omega_t \times \partial\omega_{\mathbf{X}}^N \end{array}}$$

- *Note: we find a simple symplectic form when the Lagrangian does not depend on* $d_{\mathbf{X}}\varphi$ *and* $\mathbf{\Pi}$ *:*

$$\mathbf{s} \equiv (\varphi, \mathbf{p}) \qquad \begin{cases} \mathcal{A}^*(t, \mathbf{X}, \varphi, \mathbf{p}) = \int_{\omega_t}\int_{\omega_{\mathbf{X}}} (\mathbf{p}.d_t\varphi - \mathcal{L}^*(t, \mathbf{X}, \varphi, \mathbf{p}))\, \mathrm{d}\mathbf{X}\mathrm{d}t \\ \mathcal{L}^* = \mathcal{H} = T^*(\mathbf{p}) + V(\varphi) \end{cases}$$

$$\boxed{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} d_t \begin{bmatrix} \varphi \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathcal{L}^*}{\partial \varphi} \\ \frac{\partial \mathcal{L}^*}{\partial \mathbf{p}} \end{bmatrix}} \qquad \mathbf{J}.d_t\mathbf{s} = \partial_{\mathbf{s}}\mathcal{L}^*$$

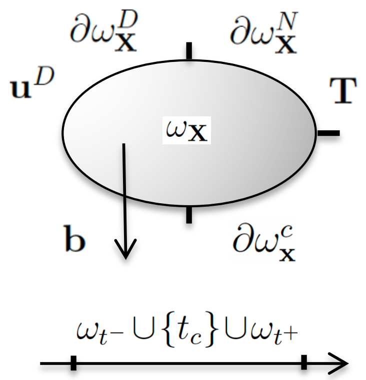
Multi-symplectic variational formalism for non-smooth elastodynamics

- *Definition of the potentially contact edge and gap:*

$$C \equiv (\mathbf{X} \in \partial\omega_{\mathbf{x}}^{c}; g_N(\mathbf{X},t) \geq 0)$$

$$g_N = (\varphi - \mathbf{X}_0).(-\mathbf{N})$$

$$\partial\omega_{\mathbf{X}} = \partial\omega_{\mathbf{X}}^{D} \cup \partial\omega_{\mathbf{X}}^{N} \cup \partial\omega_{\mathbf{x}}^{c}$$



$$\omega_{t^+} \equiv [t_c^+, t_f]$$

$$\omega_{t^-} \equiv [t_0, t_c^-]$$

- *7-field action with Lagrange multiplier for non-smooth elastodynamics:*

$$\boxed{\begin{aligned}&\widetilde{\mathcal{A}}(t, t_c, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}, \lambda)\\ &= \mathcal{A}^{*-}(t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}, \lambda) + \mathcal{A}^{*+}(t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}, \lambda) + \textstyle\int_{\partial C} \lambda(t_c).g_N(t_c)\,\mathrm{d}\mathbf{S}\end{aligned}}$$

with:

$$\begin{cases}\mathcal{A}^{*\pm}(t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi})\\ = \int_{\omega_{t^\pm}} \int_{\omega_{\mathbf{X}}} (\mathbf{p}.d_t\varphi - \mathbf{\Pi} : d_{\mathbf{x}}\varphi - \mathcal{L}^*(t, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}))\,\mathrm{d}\mathbf{X}\mathrm{d}t\\ + \int_{\omega_{t^\pm}} \int_C \lambda(t).\dot{g}_N(t)\,\mathrm{d}\mathbf{S}\mathrm{d}t\\ \mathcal{L}^* = T^*(\mathbf{p}) - \psi^*(\mathbf{\Pi}) - V_{ext}(\varphi) \quad T^*(\mathbf{p}) = \frac{1}{2}\rho^{-1}\mathbf{p}.\mathbf{p}\end{cases}$$

- *Karush-Kuhn-Tucker complementary conditions over the entire time domain:*

$$\boxed{\lambda(\mathbf{X},t).g_N(\mathbf{X},t) = 0 \;\; \lambda(\mathbf{X},t) \geq 0 \;\; g_N(\mathbf{X},t) \geq 0 \;\forall \mathbf{X} \in C \;\forall t \in [t_0, t_f]}$$

Multi-symplectic variational formalism for non-smooth elastodynamics

- *Definition of the state vector:*

$$\widetilde{\mathbf{m}} \equiv (t, t_c, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}, \lambda)$$

- *Hamilton's principle for non-smooth elastodynamics:*

$$\boxed{\begin{array}{l} \delta\widetilde{\mathcal{A}} + \int_{\omega_t}\int_{\partial\omega^N_{\mathbf{X}}} \mathbf{T}.\delta\varphi\,\mathrm{d}\mathbf{S}\mathrm{d}t = 0 \quad \forall \delta\widetilde{\mathbf{m}} \in \widetilde{\mathcal{M}}_0 \\ \widetilde{\mathcal{M}}_0 \equiv (\delta(t, t_c, \mathbf{X}, \varphi, \mathbf{p}, \mathbf{\Pi}) \in H^1(\omega_t \times \omega_{\mathbf{X}}); \delta t|_{[t_0,t_f]} = 0; \delta\mathbf{X}|_{\partial\omega_{\mathbf{X}}} = 0; .. \\ ...; \delta(\varphi, \mathbf{p}, \mathbf{\Pi})|_{[t_0,t_f]} = 0; \delta\varphi|_{\partial\omega^D_{\mathbf{X}}} = 0; \delta\lambda \in H^{\frac{1}{2}}(\omega_t \times C); \delta\lambda(t_c) > 0) \end{array}}$$

- *Definition of the 7 generators (3 horizontal, 4 vertical):*

$$\begin{aligned} \delta\widetilde{\mathcal{A}} &= \frac{\partial\widetilde{\mathcal{A}}}{\partial\widetilde{\mathbf{m}}}.\delta\widetilde{\mathbf{m}} = \delta_t\widetilde{\mathcal{A}} + \delta_{\mathbf{x}}\widetilde{\mathcal{A}} + \delta_\varphi\widetilde{\mathcal{A}} + \delta_{\mathbf{p}}\widetilde{\mathcal{A}} + \delta_{\mathbf{\Pi}}\widetilde{\mathcal{A}} + \delta_{t_c}\widetilde{\mathcal{A}} + \delta_\lambda\widetilde{\mathcal{A}} \\ &= D_t\widetilde{\mathcal{A}}.\delta t + D_{\mathbf{X}}\widetilde{\mathcal{A}}.\delta\mathbf{X} + D_\varphi\widetilde{\mathcal{A}}.\delta\varphi + D_{\mathbf{p}}\widetilde{\mathcal{A}}.\delta\mathbf{p} + D_{\mathbf{\Pi}}\widetilde{\mathcal{A}}.\delta\mathbf{\Pi} \\ &+ D_{t_c}\widetilde{\mathcal{A}}.\delta t_c + D_\lambda\widetilde{\mathcal{A}}.\delta\lambda \end{aligned}$$

Multi-symplectic variational formalism for non-smooth elastodynamics

- *Definition of state vector (without space):*

$$\boxed{\delta\widetilde{\mathbf{m}} \equiv (\delta t, \delta t_c, \delta\mathbf{X} = 0, \delta\varphi, \delta\mathbf{p}, \delta\mathbf{\Pi}, \delta\lambda)}$$

- *Calculation of the 6 corresponding generators (2 horizontal, 4 vertical):*

$$
\begin{aligned}
&\delta_t\widetilde{\mathcal{A}} + \delta_\varphi\widetilde{\mathcal{A}} + \delta_{\mathbf{p}}\widetilde{\mathcal{A}} + \delta_{\mathbf{\Pi}}\widetilde{\mathcal{A}} + \delta_{t_c}\widetilde{\mathcal{A}} + \delta_\lambda\widetilde{\mathcal{A}} = \\
&\int_{\omega_{t-}\cup\omega_{t+}} \int_{\omega_{\mathbf{X}}} (\delta\mathbf{p}.d_t\varphi + \mathbf{p}.\delta d_t\varphi - \delta\mathbf{\Pi} : d_{\mathbf{x}}\varphi - \mathbf{\Pi} : \delta d_{\mathbf{x}}\varphi)\, \mathrm{d}\mathbf{X}\mathrm{d}t \\
&- \int_{\omega_{t-}\cup\omega_{t+}} \int_{\omega_{\mathbf{X}}} \left(\tfrac{\partial\mathcal{L}^*}{\partial\varphi}.\delta\varphi + \tfrac{\partial\mathcal{L}^*}{\partial\mathbf{p}}.\delta\mathbf{p} + \tfrac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} : \delta\mathbf{\Pi}\right) \mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_{t-}\cup\omega_{t+}} \int_{C} (\delta\lambda.\dot{g}_N + \lambda.\delta\dot{g}_N)\, \mathrm{d}\mathbf{S}\mathrm{d}t \\
&+ \int_{\omega_{t-}\cup\omega_{t+}} \int_{\omega_{\mathbf{X}}} \left(\tfrac{\partial(\mathbf{p}.d_t\varphi)}{\partial d_t\varphi}.d_{tt}\varphi.\delta t + \tfrac{\partial(\mathbf{p}.d_t\varphi)}{\partial\mathbf{p}}.d_t\mathbf{p}\delta t\right) \mathrm{d}\mathbf{X}\mathrm{d}t \\
&- \int_{\omega_{t-}\cup\omega_{t+}} \int_{\omega_{\mathbf{X}}} \left(\tfrac{\partial(\mathbf{\Pi}:d_{\mathbf{x}}\varphi)}{\partial d_{\mathbf{x}}\varphi}.d_{t\mathbf{x}}\varphi.\delta t + \tfrac{\partial(\mathbf{\Pi}:d_{\mathbf{x}}\varphi)}{\partial\mathbf{\Pi}} : d_t\mathbf{\Pi}\delta t\right) \mathrm{d}\mathbf{X}\mathrm{d}t \\
&- \int_{\omega_{t-}\cup\omega_{t+}} \int_{\omega_{\mathbf{X}}} \left(\tfrac{\partial\mathcal{L}^*}{\partial t}.\delta t + \tfrac{\partial\mathcal{L}^*}{\partial\varphi}.d_t\varphi\delta t + \tfrac{\partial\mathcal{L}^*}{\partial\mathbf{p}}.d_t\mathbf{p}\delta t + \tfrac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} : d_t\mathbf{\Pi}\delta t\right) \mathrm{d}\mathbf{X}\mathrm{d}t \\
&+ \int_{\omega_{t-}\cup\omega_{t+}} \int_{C} \left(\tfrac{\partial(\lambda.\dot{g}_N)}{\partial\lambda}.d_t\lambda.\delta t + \tfrac{\partial(\lambda.\dot{g}_N)}{\partial\dot{g}_N}.d_t g_N\delta t\right) \mathrm{d}\mathbf{S}\mathrm{d}t \\
&+ \int_{\omega_{t-}\cup\omega_{t+}} \left(\int_{\omega_{\mathbf{X}}} (\mathbf{p}.d_t\varphi - \mathbf{\Pi} : d_{\mathbf{x}}\varphi - \mathcal{L}^*)\, \mathrm{d}\mathbf{X} + \int_C \lambda.\dot{g}_N\, \mathrm{d}\mathbf{S}\right) \delta(\mathrm{d}t) \\
&+ \int_C \delta\lambda(t_c).g_N(t_c)\, \mathrm{d}\mathbf{S} + \int_C \lambda(t_c).\delta g_N(t_c)\, \mathrm{d}\mathbf{S}
\end{aligned}
$$

Multi-symplectic variational formalism for non-smooth elastodynamics

- *Hamilton's principle:*

$$
\begin{aligned}
& D_t\widetilde{\mathcal{A}}.\delta t + D_\varphi\widetilde{\mathcal{A}}.\delta\varphi + D_{\mathbf{p}}\widetilde{\mathcal{A}}.\delta\mathbf{p} + D_{\mathbf{\Pi}}\widetilde{\mathcal{A}}.\delta\mathbf{\Pi} \\
& + D_{t_c}\widetilde{\mathcal{A}}.\delta t_c + D_\lambda\widetilde{\mathcal{A}}.\delta\lambda + \textstyle\int_{\omega_t}\int_{\partial\omega^N_{\mathbf{X}}} \mathbf{T}.\delta\varphi\,\mathrm{d}\mathbf{S}\mathrm{d}t = 0 \\
& \forall(\delta t, \delta t_c, \delta\varphi, \delta\mathbf{p}, \delta\mathbf{\Pi}, \delta\lambda) \in \widetilde{\mathcal{M}}_0
\end{aligned}
$$

- *Euler–Lagrange equations of non-smooth elastodynamics:*

$$
\boxed{
\begin{aligned}
& -\frac{\partial\mathcal{L}^*}{\partial\varphi} - d_t\mathbf{p} + d_{\mathbf{X}}.\mathbf{\Pi} && = 0 \quad \text{in } (\omega_{t^-}\cup\omega_{t^+})\times\omega_{\mathbf{X}} \\
& -\frac{\partial\mathcal{L}^*}{\partial\mathbf{p}} + d_t\varphi && = 0 \quad \text{if } (\omega_{t^-}\cup\omega_{t^+})\times\omega_{\mathbf{X}} \\
& -\frac{\partial\mathcal{L}^*}{\partial\mathbf{\Pi}} - d_{\mathbf{X}}\varphi && = 0 \quad \text{in } (\omega_{t^-}\cup\omega_{t^+})\times\omega_{\mathbf{X}} \\
& \mathbf{\Pi}.\mathbf{N} && = \dot\lambda\mathbf{N} \quad \text{on } (\omega_{t^-}\cup\omega_{t^+})\times\partial\omega^c_{\mathbf{X}} \\
& \mathbf{\Pi}.\mathbf{N} && = \mathbf{T} \quad \text{on } (\omega_{t^-}\cup\omega_{t^+})\times\partial\omega^N_{\mathbf{X}} \\
& \frac{\partial\mathcal{L}^*}{\partial t} + d_t(\mathbf{p}.d_t\varphi - \mathcal{L}^*) - d_{\mathbf{X}}(\mathbf{\Pi}.d_t\varphi) && = 0 \quad \text{in } (\omega_{t^-}\cup\omega_{t^+})\times\omega_{\mathbf{X}} \\
& [\mathbf{p}]^{t_c^+}_{t_c^-} && = 0 \quad \text{in } \omega_{\mathbf{X}} \\
& [\mathbf{p}.d_t\varphi - \mathcal{L}^*]^{t_c^+}_{t_c^-} && = 0 \quad \text{in } \omega_{\mathbf{X}} \\
& \lambda(t_c).\dot g_N(t_c) && = 0 \quad \text{on } \partial\omega^c_{\mathbf{X}} \\
& g_N(t_c) && = 0 \quad \text{on } \partial\omega^c_{\mathbf{X}} \\
& +\text{KKT conditions}
\end{aligned}
}
$$

- *The KKT conditions combined with the two previous contact conditions are written:*

$$\begin{cases} \lambda(\mathbf{X},t).g_N(\mathbf{X},t)=0 \quad \lambda(\mathbf{X},t)\geq 0 \quad g_N(\mathbf{X},t)\geq 0 \; \forall \mathbf{X}\in C \; \forall t\in[t_0,t_f] \\ \lambda(t_c).\dot{g}_N(t_c) \qquad\qquad = 0 \quad \text{on } \partial\omega^c_{\mathbf{x}} \\ g_N(t_c) \qquad\qquad = 0 \quad \text{on } \partial\omega^c_{\mathbf{x}} \end{cases}$$

- *These conditions are strictly equivalent to the Hertz-Signorini-Moreau (HSM) conditions:*

$$\boxed{\begin{cases} g_N(t)\geq 0 \\ \lambda(t)\geq 0 \\ \lambda(t).g_N(t)=0 \\ \lambda(t_c).\dot{g}_N(t_c)=0 \\ g_N(t_c)=0 \end{cases} \iff \begin{cases} \text{if} \quad g_N(t)>0 \quad \text{then} \quad \lambda(t)=0 \\ \text{if} \quad g_N(t)=0 \quad \text{then} \quad \begin{cases} \dot{g}_N(t)\geq 0 \\ \lambda(t)\geq 0 \\ \lambda(t).\dot{g}_N(t)=0 \end{cases} \end{cases}}$$

Multi-symplectic variational formalism for non-smooth elastodynamics

- *From the first derivatives of the Lagrangian, the Euler-Lagrange equations of non-smooth elastodynamics are obtained as part of a mixed formulation, to which we associate the HSM conditions:*

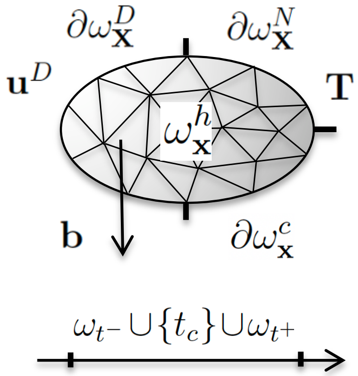
$$
\boxed{
\begin{array}{lll}
\mathbf{b} - d_t\mathbf{p} + d_{\mathbf{x}}.\mathbf{\Pi} & = 0 & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega_{\mathbf{X}} \\
-\rho^{-1}\mathbf{p} + d_t\varphi & = 0 & \text{if } (\omega_{t^-} \cup \omega_{t^+}) \times \omega_{\mathbf{X}} \\
\mathbf{F} - d_{\mathbf{x}}\varphi & = 0 & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega_{\mathbf{X}} \\
\mathbf{\Pi}.\mathbf{N} & = \dot{\lambda}.\mathbf{N} & \text{on } (\omega_{t^-} \cup \omega_{t^+}) \times \partial\omega_{\mathbf{X}}^c \\
\mathbf{\Pi}.\mathbf{N} & = \mathbf{T} & \text{on } (\omega_{t^-} \cup \omega_{t^+}) \times \partial\omega_{\mathbf{X}}^N \\
d_t(T^*(\mathbf{p}) + \psi^*(\mathbf{\Pi})) - d_{\mathbf{x}}(\mathbf{\Pi}.d_t\varphi) & = 0 & \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega_{\mathbf{X}} \\
[\mathbf{p}]_{t_c^-}^{t_c^+} & = 0 & \text{in } \omega_{\mathbf{X}} \\
[T^*(\mathbf{p})]_{t_c^-}^{t_c^+} & = 0 & \text{in } \omega_{\mathbf{X}} \\
+\text{HSM conditions} & &
\end{array}
}
$$

- *Similarly, the expression of the equation for the conservation of configurational forces (mixed formulation) can be calculated from the horizontal space generator* $D_{\mathbf{X}}\tilde{\mathcal{A}}$ *:*

$$\frac{\partial \mathcal{L}^*}{\partial \mathbf{X}} + d_{\mathbf{x}}.(-\mathbf{\Pi}.d_{\mathbf{x}}\varphi - \mathcal{L}^*\mathbf{I}) + d_t(\mathbf{p}.d_{\mathbf{x}}\varphi) = 0$$

Space semi-discretized variational formulation for non-smooth elastodynamics

- *Mesh of geometry:*



Node coordinates:

$\mathbf{X}_j$

- *Spatial discretization by the finite element method:*

$$\boxed{\begin{aligned}
\varphi(t,\mathbf{X}) &\simeq \varphi^h(t,\mathbf{X}) = \sum_{i=1}^{n} u_i(t).\phi_i^u(\mathbf{X}) \\
\mathbf{p}(t,\mathbf{X}) &\simeq \mathbf{p}^h(t,\mathbf{X}) = \sum_{i=1}^{n} p_i(t).\phi_i^p(\mathbf{X}) \\
\mathbf{\Pi}(t,\mathbf{X}) &\simeq \mathbf{\Pi}^h(t,\mathbf{X}) = \sum_{i=1}^{n} s_i(t).\phi_i^s(\mathbf{X}) \\
\lambda(t,\mathbf{X}) &\simeq \lambda^h(t,\mathbf{X}) = \sum_{i=1}^{p} \lambda_i(t).\phi_i^\lambda(\mathbf{X})
\end{aligned}}$$

- *Partition of unity and compact support:* $\sum_{i=1}^{n} \phi_i^u(\mathbf{X}) = 1$ $\quad \phi_i^u(\mathbf{X}_j) = \delta_{ij}$

Space semi-discretized variational formulation for non-smooth elastodynamics

- *Space semi-discretized multi-field action for non-smooth elastodynamics:*

$$\boxed{\begin{aligned}&\widetilde{\mathcal{A}}^h(t,t_c,\mathbf{U},\mathbf{P},\mathbf{F}_{int},\boldsymbol{\Lambda})\\&=\int_{\omega_{t^-}\cup\omega_{t^+}}\left(\mathbf{P}^T.\dot{\mathbf{U}}-\mathbf{F}_{int}^T.\mathbf{U}-\mathcal{L}^{h*}+\boldsymbol{\Lambda}^T.\dot{\mathbf{g}}_N\right)\,\mathrm{d}t+\Lambda(t_c)^T.\mathbf{g}_N(t_c)\\&+\text{KKT conditions}\\&\text{with}\quad \mathcal{L}^{h*}=T^{h*}(\mathbf{P})-\Psi^{h*}(\mathbf{F}_{int})-\mathbf{F}_{ext}^T.\mathbf{U}\end{aligned}}$$

- *Spatial discretization of the different terms:*
  - $\int_{\omega_\mathbf{X}}\mathbf{p}.d_t\varphi\,\mathrm{d}\mathbf{X}\simeq\int_{\omega_\mathbf{X}^h}\left(\sum_{i=1}^n p_i(t).\phi_i^p(\mathbf{X})\right).d_t\left(\sum_{j=1}^n u_j(t).\phi_j^u(\mathbf{X})\right)\,\mathrm{d}\mathbf{X}$
    $=\sum_{i=1}^n\sum_{i=1}^n p_i(t)\left(\int_{\omega_\mathbf{X}^h}\phi_i^p.\phi_j^u\,\mathrm{d}\mathbf{X}\right)d_t u_j(t)=\mathbf{P}^T.\dot{\mathbf{U}}$
  - $\int_{\omega_\mathbf{X}}\boldsymbol{\Pi}:d_\mathbf{x}\varphi\,\mathrm{d}\mathbf{X}\simeq\sum_{i=1}^n\sum_{j=1}^n s_i(t).\left(\int_{\omega_\mathbf{X}^h}\phi_i^s:d_\mathbf{x}\phi_j^u\,\mathrm{d}\mathbf{X}\right)u_j(t)=\mathbf{F}_{int}^T.\mathbf{U}$
  - $\int_{\omega_\mathbf{X}}\mathbf{b}.\varphi\,\mathrm{d}\mathbf{X}\simeq\sum_{i=1}^n\left(\int_{\omega_\mathbf{X}^h}\mathbf{b}.\phi_i^u(\mathbf{X})\,\mathrm{d}\mathbf{X}\right)u_i(t)=\mathbf{F}_{ext}^T.\mathbf{U}$
  - $\int_C\lambda(t).\dot{g}_N(t)\,\mathrm{d}\mathbf{S}\simeq\sum_{i=1}^p\sum_{j=1}^p\lambda_i(t).(\int_C\phi_i^\lambda.\phi_j^\lambda\,\mathrm{d}\mathbf{S})\dot{g}_{N_j}(t)=\boldsymbol{\Lambda}^T.\dot{\mathbf{g}}_N$
  - $\int_C\dot{\lambda}.\delta g_N\,\mathrm{d}\mathbf{S}=\int_C\dot{\lambda}.d_\varphi g_N\delta\varphi\,\mathrm{d}\mathbf{S}$
    $\simeq\sum_{i=1}^p\sum_{j=1}^n\dot{\lambda}_i.(\int_C\phi_i^\lambda.\phi_j^u.(-\mathbf{N})\,\mathrm{d}\mathbf{S})\delta U_j=\dot{\boldsymbol{\Lambda}}^T.(-\mathbf{L}).\delta\mathbf{U}=\dot{\boldsymbol{\Lambda}}^T.\delta\mathbf{g}_N$

Space semi-discretized variational formulation for non-smooth elastodynamics

- *Hamilton's principle semi-discretized in space for non-smooth elastodynamics:*

$$\boxed{\begin{aligned}&\delta\widetilde{\mathcal{A}}^h + \textstyle\int_{\omega_t} {\mathbf{F}^s_{ext}}^T.\delta\mathbf{U}\,\mathrm{d}t = 0 \quad \forall\delta\widetilde{\mathbf{m}}^h \in \widetilde{\mathcal{M}}^h_0\\ &\widetilde{\mathcal{M}}^h_0 \equiv (\delta(\mathbf{U},\mathbf{P},\mathbf{F}_{int}) \in H^1(\omega_t\times\omega^h_{\mathbf{x}});\delta t|_{[t_0,t_f]} = 0;\ldots\\ &\ldots\delta(\mathbf{U},\mathbf{P},\mathbf{F}_{int})|_{[t_0,t_f]} = 0;\delta\mathbf{U}|_{\partial\omega^D_{\mathbf{x}}} = 0;\delta\mathbf{\Lambda}\in H^{\frac{1}{2}}(\omega_t\times C);\ldots\\ &\ldots\delta\Lambda^i(t_c) > 0, i\in\{1,\ldots p\})\end{aligned}}$$

with: $$\textstyle\int_{\partial\omega^N_{\mathbf{x}}} \mathbf{T}.\delta\varphi\,\mathrm{d}\mathbf{S} \simeq \sum_{i=1}^{n}(\int_{\partial\omega^N_{\mathbf{x}}}\mathbf{T}.\phi_i\,\mathrm{d}\mathbf{S}).\delta U_i = {\mathbf{F}^s_{ext}}^T.\delta\mathbf{U}$$

Space semi-discretized variational formulation for non-smooth elastodynamics

- Definition of the 6 space-separated generators (2 horizontal and 4 vertical):

$$\delta \widetilde{\mathcal{A}}^h = \frac{\partial \widetilde{\mathcal{A}}^h}{\partial \widetilde{\mathbf{m}}^h} . \delta \widetilde{\mathbf{m}}^h = \delta_t \widetilde{\mathcal{A}}^h + \delta_{t_c} \widetilde{\mathcal{A}}^h + \delta_{\mathbf{U}} \widetilde{\mathcal{A}}^h + \delta_{\mathbf{P}} \widetilde{\mathcal{A}}^h + \delta_{\mathbf{F}_{int}} \widetilde{\mathcal{A}}^h + \delta_{\mathbf{\Lambda}} \widetilde{\mathcal{A}}^h$$
$$= D_t \widetilde{\mathcal{A}}^h . \delta t + D_{t_c} \widetilde{\mathcal{A}}^h . \delta t_c + D_{\mathbf{U}} \widetilde{\mathcal{A}}^h . \delta \mathbf{U} + D_{\mathbf{P}} \widetilde{\mathcal{A}}^h . \delta \mathbf{P} + D_{\mathbf{F}_{int}} \widetilde{\mathcal{A}}^h . \delta \mathbf{F}_{int}$$
$$+ D_{\mathbf{\Lambda}} \widetilde{\mathcal{A}} . \delta \mathbf{\Lambda}$$

- Calculation of the 6 corresponding generators

$$\delta \widetilde{\mathcal{A}}^h = \int_{\omega_{t-} \cup \omega_{t+}} \left(\delta \mathbf{P}^T . \dot{\mathbf{U}} + \mathbf{P}^T . \delta \dot{\mathbf{U}} - \delta \mathbf{F}_{int}^T . \mathbf{U} - \mathbf{F}_{int}^T . \delta \mathbf{U} \right) \mathrm{d}t$$
$$+ \int_{\omega_{t-} \cup \omega_{t+}} \left(-(\frac{\partial \mathcal{L}^{h*}}{\partial \mathbf{U}})^T . \delta \mathbf{U} - (\frac{\partial \mathcal{L}^{h*}}{\partial \mathbf{P}})^T . \delta \mathbf{P} - (\frac{\partial \mathcal{L}^{h*}}{\partial \mathbf{F}_{int}})^T . \delta \mathbf{F}_{int} \right) \mathrm{d}t$$
$$+ \int_{\omega_{t-} \cup \omega_{t+}} \left(\delta \mathbf{\Lambda}^T . \dot{\mathbf{g}}_N + \mathbf{\Lambda}^T . \delta \dot{\mathbf{g}}_N \right) \mathrm{d}t$$
$$+ \int_{\omega_{t-} \cup \omega_{t+}} \left((\frac{\partial (\mathbf{P}^T . \dot{\mathbf{U}})}{\partial \mathbf{P}})^T . \dot{\mathbf{P}} \delta t + (\frac{\partial (\mathbf{P}^T . \dot{\mathbf{U}})}{\partial \dot{\mathbf{U}}})^T . \ddot{\mathbf{U}} \delta t \right) \mathrm{d}t$$
$$+ \int_{\omega_{t-} \cup \omega_{t+}} \left((\frac{\partial ({\mathbf{F}_{int}}^T . \dot{\mathbf{U}})}{\partial \mathbf{F}_{int}})^T . \dot{\mathbf{F}}_{int} \delta t + (\frac{\partial ({\mathbf{F}_{int}}^T . \mathbf{U})}{\partial \mathbf{U}})^T . \dot{\mathbf{U}} \delta t \right) \mathrm{d}t$$
$$+ \int_{\omega_{t-} \cup \omega_{t+}} \left(-(\frac{\partial \mathcal{L}^{h*}}{\partial t})^T \delta t - (\frac{\partial \mathcal{L}^{h*}}{\partial \mathbf{U}})^T . \dot{\mathbf{U}} \delta t - (\frac{\partial \mathcal{L}^{h*}}{\partial \mathbf{P}})^T . \dot{\mathbf{P}} \delta t - (\frac{\partial \mathcal{L}^{h*}}{\partial \mathbf{F}_{int}})^T . \dot{\mathbf{F}}_{int} \delta t \right) \mathrm{d}t$$
$$+ \int_{\omega_{t-} \cup \omega_{t+}} \left((\frac{\partial (\mathbf{\Lambda}^T . \dot{\mathbf{g}}_N)}{\partial \mathbf{\Lambda}})^T . \dot{\mathbf{\Lambda}} \delta t + (\frac{\partial (\mathbf{\Lambda}^T . \dot{\mathbf{g}}_N)}{\partial \dot{\mathbf{g}}_N})^T . \ddot{\mathbf{g}}_N \delta t \right) \mathrm{d}t$$
$$+ \int_{\omega_{t-} \cup \omega_{t+}} \left(\mathbf{P}^T . \dot{\mathbf{U}} - \mathbf{F}_{int}^T . \mathbf{U} - \mathcal{L}^{h*} + \mathbf{\Lambda}^T . \dot{\mathbf{g}}_N) \right) \delta(\mathrm{d}t)$$
$$+ \delta \mathbf{\Lambda}(t_c)^T . \mathbf{g}_N(t_c) + \mathbf{\Lambda}(t_c)^T . \delta \mathbf{g}_N(t_c)$$

Space semi-discretized variational formulation for non-smooth elastodynamics

- *Euler–Lagrange equations semi-discretized in space for non-smooth elastodynamics:*

$$
\boxed{
\begin{aligned}
&\frac{\partial \mathcal{L}^{h*}}{\partial \mathbf{U}} + \dot{\mathbf{P}} + \mathbf{F}_{int} &&= \mathbf{F}^s_{ext} + \mathbf{L}^T \dot{\mathbf{\Lambda}} \quad \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
&-\frac{\partial \mathcal{L}^*}{\partial \mathbf{P}} + \dot{\mathbf{U}} &&= 0 \quad \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
&-\frac{\partial \mathcal{L}^*}{\partial \mathbf{F}_{int}} - \mathbf{U} &&= 0 \quad \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
&-\frac{\partial \mathcal{L}^{h*}}{\partial t} + d_t(\mathcal{L}^{h*} + \mathbf{F}^T_{int}\mathbf{U}) &&= (\mathbf{L}^T \dot{\mathbf{\Lambda}})^T . \dot{\mathbf{U}} + {\mathbf{F}^s_{ext}}^T . \dot{\mathbf{U}} \quad \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
&[\mathbf{P}]^{t_c^+}_{t_c^-} &&= 0 \quad \text{in } \omega^h_{\mathbf{x}} \\
&[\mathbf{P}^T . \dot{\mathbf{U}} - \mathcal{L}^{h*}]^{t_c^+}_{t_c^-} &&= 0 \quad \text{in } \omega^h_{\mathbf{x}} \\
&\mathbf{\Lambda}(t_c)^T . \dot{\mathbf{g}}_N(t_c) &&= 0 \quad \text{on } \partial\omega^c_{\mathbf{x}} \\
&\mathbf{g}_N(t_c) &&= 0 \quad \text{on } \partial\omega^c_{\mathbf{x}} \\
&+\text{KKT conditions}
\end{aligned}
}
$$

Space semi-discretized variational formulation for non-smooth elastodynamics

- *Expressions of the first derivatives of the space-semi-discretized Lagrangian:*

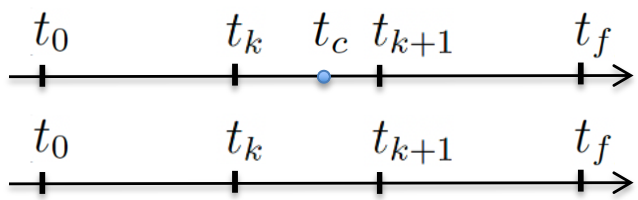
$$\frac{\partial \mathcal{L}^{h*}}{\partial \mathbf{U}} = -\mathbf{F}_{ext}$$
$$\frac{\partial \mathcal{L}^{h*}}{\partial \mathbf{P}} = \frac{\partial T^*}{\partial \mathbf{P}} = \mathbf{M}^{-1}\mathbf{P}$$
$$\frac{\partial \mathcal{L}^{h*}}{\partial \mathbf{F}_{int}} = -\frac{\partial \psi^*}{\partial \mathbf{F}_{int}} = -\mathbf{K}_{tang}^{-1}\mathbf{F}_{int}$$

- *Euler–Lagrange equations semi-discretized in space for non-smooth elastodynamics:*

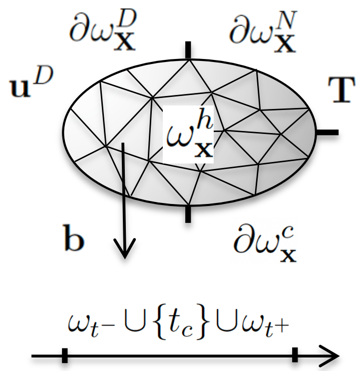
$$\boxed{\begin{array}{ll}
\dot{\mathbf{P}} + \mathbf{F}_{int} & = \mathbf{F}_{ext} + \mathbf{F}^{s}_{ext} + \mathbf{L}^T\dot{\mathbf{\Lambda}} \quad \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
-\mathbf{M}^{-1}\mathbf{P} + \dot{\mathbf{U}} & = 0 \quad \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
\mathbf{K}_{tang}^{-1}\mathbf{F}_{int} - \mathbf{U} & = 0 \quad \text{in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
d_t(T^{h^*}(\mathbf{P}) + \psi^*(\mathbf{F}_{int})) & = (\mathbf{F}_{ext} + \mathbf{F}^{s}_{ext} + \mathbf{L}^T\dot{\mathbf{\Lambda}})^T.\dot{\mathbf{U}} \text{ in } (\omega_{t^-} \cup \omega_{t^+}) \times \omega^h_{\mathbf{x}} \\
[\mathbf{P}]_{t_c^-}^{t_c^+} & = 0 \text{ in } \times \omega^h_{\mathbf{x}} \\
[\frac{1}{2}\mathbf{P}^T.\mathbf{M}^{-1}.\mathbf{P}]_{t_c^-}^{t_c^+} & = 0 \text{ in } \times \omega^h_{\mathbf{x}} \\
+\text{HSM conditions} &
\end{array}}$$

Variational formulation discretized in space and time for non-smooth elastodynamics

- *Two families of time integrators in non-smooth elastodynamics:*
  - *Event driven*
  - *Time stepping*



$$\tau = t_{k+1} - t_k$$



Unlike the continuous-time formulation, and following Moreau's ideas, ***the instant of impact does not have to be detected****. Indeed, the impact is here taken in the weak sense. In other words, many impacts can occur during a time step. However, it is not necessary to detect them precisely. Here, a main difference must be made between "event-driven" integrators (detection of the event of an impact) and "time-stepping" time integrators (no detection of the event of an impact in the strong sense).* ***Here, the "time-stepping" approach will be favored in order to possibly allow an infinity of impact events in a finite time (Zeno's paradox)****. Therefore, the time approximation no longer contains the instant of impact t_c as a variable of the action.*

Variational formulation discretized in space and time for non-smooth elastodynamics

- *Multi-field action discretized in space and time for nonsmooth elastodynamics:*

$$\overset{\sim}{\mathcal{A}}^{h}(t) \simeq \sum_{k=0}^{N-1} \overset{\sim}{\mathcal{A}}^{h\tau}(t_{k+1}, \mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k+1}, \mathbf{\Lambda}_{k+\frac{1}{2}})$$

$$\boxed{\begin{aligned}&\overset{\sim}{\mathcal{A}}^{h\tau}(t_{k+1}, \mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k+1}, \mathbf{\Lambda}_{k+\frac{1}{2}}))\\ &= \tau \mathbf{P}^T_{k+\frac{1}{2}}.\frac{(\mathbf{U}_{k+1}-\mathbf{U}_k)}{\tau} - \frac{\tau}{2}(\mathbf{F}_{int\,k}^{T}.\mathbf{U}_k + \mathbf{F}_{int\,k+1}^{T}.\mathbf{U}_{k+1})\\ &-\frac{\tau}{2}(\mathcal{L}^{h\tau *}(\mathbf{U}_k, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k}) + \mathcal{L}^{h\tau *}(\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k+1}))\\ &+\tau \mathbf{\Lambda}^T_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}}\\ &+ \text{HSM conditions}\end{aligned}}$$

with:

$$\mathcal{L}^{h\tau *}(\mathbf{U}_k, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k}) = T^{h\tau *}(\mathbf{P}_{k+\frac{1}{2}}) - \Psi^{h\tau *}(\mathbf{F}_{int\,k}) - \mathbf{F}^T_{ext\,k}.\mathbf{U}_k$$

$$\begin{cases} \text{if} \quad g^i_{N\,k+1} > 0 \quad \text{then} \quad \Lambda^i_{k+\frac{3}{2}} = 0 \\ \text{if} \quad g^i_{N\,k+1} \leq 0 \quad \text{then} \quad \begin{cases} \dot{g}^i_{N_{k+\frac{3}{2}}} \geq 0 \\ \Lambda^i_{k+\frac{3}{2}} \geq 0 \\ \Lambda^T_{k+\frac{3}{2}}.\dot{g}^i_{N_{k+\frac{3}{2}}} = 0 \end{cases} \end{cases} \qquad i \in \{1, \ldots p\}$$

Variational formulation discretized in space and time for non-smooth elastodynamics

- State vector:

$$\boxed{\overset{\sim}{\mathbf{m}}^{h\tau} \equiv (t_{k+1}, \mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k+1}, \mathbf{\Lambda}_{k+\frac{1}{2}})}$$

- Discrete space-time Hamilton's principle:

$$\boxed{\begin{aligned} &\delta \overset{\sim}{\mathcal{A}}^{h\tau} + \sum_{k=0}^{N-1} \tau {\mathbf{F}^{s}_{ext\,k+1}}^{T} . \delta \mathbf{U}_{k+1} = 0 \quad \forall \delta \overset{\sim}{\mathbf{m}}^{h\tau} \in \overset{\sim}{\mathcal{M}}_0^{h\tau} \\ &\overset{\sim}{\mathcal{M}}_0^{h\tau} \equiv (\delta \overset{\sim}{\mathbf{m}}^{h\tau}|_{\partial\omega_t} = 0; \delta \mathbf{U}|_{\partial\omega_{\mathbf{X}}^{D}} = 0; \delta \mathbf{\Lambda} = 0) \end{aligned}}$$

- Discrete space-time generators (1 horizontal and 4 vertical):

$$\begin{aligned} \delta \overset{\sim}{\mathcal{A}}^{h\tau} &= \frac{\partial \overset{\sim}{\mathcal{A}}^{h\tau}}{\partial \overset{\sim}{\mathbf{m}}^{h\tau}} . \delta \overset{\sim}{\mathbf{m}}^{h\tau} = \delta_t \overset{\sim}{\mathcal{A}}^{h\tau} + \delta_{\mathbf{U}} \overset{\sim}{\mathcal{A}}^{h\tau} + \delta_{\mathbf{P}} \overset{\sim}{\mathcal{A}}^{h\tau} + \delta_{\mathbf{F}_{int}} \overset{\sim}{\mathcal{A}}^{h\tau} + \delta_{\mathbf{\Lambda}} \overset{\sim}{\mathcal{A}}^{h\tau} \\ &= D_t \overset{\sim}{\mathcal{A}}^{h\tau} . \delta t_{k+1} + D_{\mathbf{U}} \overset{\sim}{\mathcal{A}}^{h\tau} . \delta \mathbf{U}_{k+1} + D_{\mathbf{P}} \overset{\sim}{\mathcal{A}}^{h\tau} . \delta \mathbf{P}_{k+\frac{1}{2}} \\ &+ D_{\mathbf{F}_{int}} \overset{\sim}{\mathcal{A}}^{h\tau} . \delta \mathbf{F}_{int\,k+1} + D_{\mathbf{\Lambda}} \overset{\sim}{\mathcal{A}}^{h\tau} . \delta \mathbf{\Lambda}_{k+\frac{1}{2}} \end{aligned}$$

Variational formulation discretized in space and time for non-smooth elastodynamics

- Calculation of the 5 corresponding generators (1 horizontal and 4 vertical):

$$
\begin{aligned}
&\sum_{k=0}^{N-1}(\delta\mathbf{P}^T_{k+\frac{1}{2}}.(\mathbf{U}_{k+1}-\mathbf{U}_k)+\mathbf{P}^T_{k+\frac{1}{2}}.\delta\mathbf{U}_{k+1}-\mathbf{P}^T_{k+\frac{1}{2}}.\delta\mathbf{U}_k\\
&-\tfrac{\delta\tau}{2}(\mathbf{F}_{int\,k}^{\ T}.\mathbf{U}_k+\mathbf{F}_{int\,k+1}^{\ T}.\mathbf{U}_{k+1})\\
&-\tfrac{\tau}{2}(\delta\mathbf{F}_{int\,k}^{\ T}.\mathbf{U}_k+\mathbf{F}_{int\,k}^{\ T}.\delta\mathbf{U}_k+\delta\mathbf{F}_{int\,k+1}^{\ T}.\mathbf{U}_{k+1}+\mathbf{F}_{int\,k+1}^{\ T}.\delta\mathbf{U}_{k+1})\\
&-\tfrac{\delta\tau}{2}(\mathcal{L}^{h\tau *}(\mathbf{U}_k,\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{int\,k})+\mathcal{L}^{h\tau *}(\mathbf{U}_{k+1},\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{int\,k+1}))\\
&-\tfrac{\tau}{2}(\tfrac{\partial\mathcal{L}^{h\tau *}}{\partial\mathbf{U}_k}(\mathbf{U}_k,\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{int\,k})\delta\mathbf{U}_k+\tfrac{\partial\mathcal{L}^{h\tau *}}{\partial\mathbf{P}_{k+\frac{1}{2}}}(\mathbf{U}_k,\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{int\,k})\delta\mathbf{P}_{k+\frac{1}{2}}+\ldots\\
&\ldots+\tfrac{\partial\mathcal{L}^{h\tau *}}{\partial\mathbf{F}_{int\,k}}(\mathbf{U}_k,\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{int\,k})\delta\mathbf{F}_{int\,k})\\
&-\tfrac{\tau}{2}(\tfrac{\partial\mathcal{L}^{h\tau *}}{\partial\mathbf{U}_{k+1}}(\mathbf{U}_{k+1},\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{int\,k+1})\delta\mathbf{U}_{k+1}+\ldots\\
&\ldots+\tfrac{\partial\mathcal{L}^{h\tau *}}{\partial\mathbf{P}_{k+\frac{1}{2}}}(\mathbf{U}_{k+1},\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{int\,k+1})\delta\mathbf{P}_{k+\frac{1}{2}}+\ldots\\
&\ldots+\tfrac{\partial\mathcal{L}^{h\tau *}}{\partial\mathbf{F}_{int\,k+1}}(\mathbf{U}_{k+1},\mathbf{P}_{k+\frac{1}{2}},\mathbf{F}_{int\,k+1})\delta\mathbf{F}_{int\,k+1})\\
&+\delta\tau\boldsymbol{\Lambda}^T_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}}+\tau\delta\boldsymbol{\Lambda}^T_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}}+\tau\boldsymbol{\Lambda}^T_{k+\frac{1}{2}}.\delta\dot{\mathbf{g}}_{N_{k+\frac{3}{2}}})\\
&+\tau\mathbf{F}^{s\ \ T}_{ext\,k+1}.\delta\mathbf{U}_{k+1})=0\quad\forall\delta\tilde{\mathbf{m}}^{h\tau}\in\tilde{\mathcal{M}}^{h\tau}_0\\
&+\ \text{HSM conditions}
\end{aligned}
$$

Variational formulation discretized in space and time for non-smooth elastodynamics

- *Vertical generator associated with linear momentum:*

$$\delta \mathbf{P}_{k+\frac{1}{2}} \qquad D_{\mathbf{P}} \overset{\sim}{\mathcal{A}}^{h\tau} = (\mathbf{U}_{k+1} - \mathbf{U}_k) - \frac{\tau}{2} \frac{\partial \mathcal{L}^{h\tau *}}{\partial \mathbf{P}_{k+\frac{1}{2}}} (\mathbf{U}_k, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k}) + \ldots$$

$$\ldots - \frac{\tau}{2} \frac{\partial \mathcal{L}^{h\tau *}}{\partial \mathbf{P}_{k+\frac{1}{2}}} (\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k+1}) = 0$$

$$\boxed{\mathbf{U}_{k+1} = \mathbf{U}_k + \tau \mathbf{M}^{-1} \mathbf{P}_{k+\frac{1}{2}}}$$

- *Vertical generator associated with displacement (space-time discretized equation of motion):*

$$\delta \mathbf{U}_{k+1} \qquad D_{\mathbf{U}} \overset{\sim}{\mathcal{A}}^{h\tau} = -\mathbf{P}_{k+\frac{3}{2}} + \mathbf{P}_{k+\frac{1}{2}} - \tau \mathbf{F}_{int\,k+1}$$

$$-\frac{\tau}{2} \left(\frac{\partial \mathcal{L}^{h\tau *}}{\partial \mathbf{U}_{k+1}} (\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k+1}) - \frac{\tau}{2} \frac{\partial \mathcal{L}^{h\tau *}}{\partial \mathbf{U}_{k+1}} (\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{3}{2}}, \mathbf{F}_{int\,k+1}) \right)$$

$$+\mathbf{L}^T \mathbf{\Lambda}_{k+\frac{3}{2}} - \mathbf{L}^T \mathbf{\Lambda}_{k+\frac{1}{2}} + \tau \mathbf{F}^s_{ext\,k+1} = 0$$

with: $\delta \dot{\mathbf{g}}_{N\,k+\frac{1}{2}} = -\mathbf{L} \delta \dot{\mathbf{U}}_{k+\frac{1}{2}}$

$$\boxed{\mathbf{P}_{k+\frac{3}{2}} = \mathbf{P}_{k+\frac{1}{2}} + \tau (\mathbf{F}_{ext\,k+1} + \mathbf{F}^s_{ext\,k+1} - \mathbf{F}_{int\,k+1}) + \mathbf{L}^T (\mathbf{\Lambda}_{k+\frac{3}{2}} - \mathbf{\Lambda}_{k+\frac{1}{2}})}$$

Variational formulation discretized in space and time for non-smooth elastodynamics

- *Vertical generator associated with internal forces:*

$\delta \mathbf{F}_{int\,k+1}$

$$D_{\mathbf{F}_{int}} \overset{\sim h\tau}{\mathcal{A}} = \mathbf{U}_{k+1} + \frac{1}{2}\left(\frac{\partial \mathcal{L}^{h\tau *}}{\partial \mathbf{F}_{int\,k+1}}(\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k+1}) + \frac{\partial \mathcal{L}^{h\tau *}}{\partial \mathbf{F}_{int\,k+1}}(\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{3}{2}}, \mathbf{F}_{int\,k+1})\right) = 0$$

$$\boxed{\mathbf{U}_{k+1} - \mathbf{K}_{tang}^{-1}\mathbf{F}_{int\,k+1} = 0}$$

- *Horizontal generator associated with time (space-time discretized energy balance):*

δt_{k+1}

$$\boxed{\begin{aligned}&((T^{h\tau *}(\mathbf{P}_{k+\frac{3}{2}}) + \tfrac{1}{2}(\Psi^{h\tau *}(\mathbf{F}_{int\,k+1}) + \Psi^{h\tau *}(\mathbf{F}_{int\,k+2}))\\ &-(T^{h\tau *}(\mathbf{P}_{k+\frac{1}{2}}) + \tfrac{1}{2}(\Psi^{h\tau *}(\mathbf{F}_{int\,k}) + \Psi^{h\tau *}(\mathbf{F}_{int\,k+1})))\\ &= \tfrac{1}{2}((\mathbf{F}^T_{ext\,k+1}.\mathbf{U}_{k+1} + \mathbf{F}^T_{ext\,k+2}.\mathbf{U}_{k+2}) - (\mathbf{F}^T_{ext\,k}.\mathbf{U}_{k} + \mathbf{F}^T_{ext\,k+1}.\mathbf{U}_{k+1}))\\ &-(\mathbf{\Lambda}^T_{k+\frac{3}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{3}{2}}} - \mathbf{\Lambda}^T_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}})\end{aligned}}$$

Variational formulation discretized in space and time for non-smooth elastodynamics

- Discretized Euler–Lagrange equations for non-smooth elastodynamics:

$$\boxed{\begin{aligned}&\mathbf{U}_{k+1} = \mathbf{U}_k + \tau \mathbf{M}^{-1}\mathbf{P}_{k+\frac{1}{2}}\\ &\mathbf{U}_{k+1} - \mathbf{K}_{tang}^{-1}\mathbf{F}_{int\,k+1} = 0\\ &\mathbf{P}_{k+\frac{3}{2}} = \mathbf{P}_{k+\frac{1}{2}} + \tau(\mathbf{F}_{ext\,k+1} + \mathbf{F}^s_{ext\,k+1} - \mathbf{F}_{int\,k+1}) + \mathbf{L}^T(\mathbf{\Lambda}_{k+\frac{3}{2}} - \mathbf{\Lambda}_{k+\frac{1}{2}})\\ &+ \text{HSM conditions}\end{aligned}}$$

with:

$$\mathbf{g}_{N\,k+1} = -(\mathbf{L}_{k+1}\mathbf{U}_{k+1} - \mathbf{X}_0.\mathbf{N})$$

$$\dot{\mathbf{g}}_{N_{free}} = -\mathbf{L}\mathbf{M}^{-1}\mathbf{P}_{free} \qquad \dot{\mathbf{g}}_{N_{k+\frac{3}{2}}} = -\mathbf{L}\mathbf{M}^{-1}\mathbf{P}_{k+\frac{3}{2}}$$

$$\mathbf{P}_{free} = \mathbf{P}_{k+\frac{1}{2}} + \tau(\mathbf{F}_{ext\,k+1} + \mathbf{F}^s_{ext\,k+1} - \mathbf{F}_{int\,k+1}) - \mathbf{L}^T\mathbf{\Lambda}_{k+\frac{1}{2}}$$

- Solving / LCP type problem ($\mathbf{M}$ and $\mathbf{H}$ are diagonal):

$$\boxed{\left\{\begin{aligned}&\mathbf{H}\mathbf{\Lambda}_{k+\frac{3}{2}} = \dot{\mathbf{g}}_{N_{k+\frac{3}{2}}} - \dot{\mathbf{g}}_{N_{free}} \qquad\qquad \mathbf{H} = \mathbf{L}\mathbf{M}^{-1}\mathbf{L}^T\\ &\text{if } \; g^i_{N\,k+1} > 0 \; \text{ then } \; \Lambda^i_{k+\frac{3}{2}} = 0\\ &\text{if } \; g^i_{N\,k+1} \leq 0 \; \text{ then } \left\{\begin{aligned}\dot{g}^i_{N_{k+\frac{3}{2}}} &\geq 0 \qquad i \in \{1, ...p\}\\ \Lambda^i_{k+\frac{3}{2}} &\geq 0\\ \Lambda^T_{k+\frac{3}{2}}.\dot{g}^i_{N_{k+\frac{3}{2}}} &= 0\end{aligned}\right.\end{aligned}\right.}$$

Steklov-Poincaré operator (Delassus)

Variational formulation discretized in space and time for non-smooth elastodynamics

- *Explicit algorithm of the LCP problem (explicit CD-Lagrange variational integrator):*

---

1 - calculate $\mathbf{U}_{k+1}$ with $\mathbf{U}_{k+1} = \mathbf{U}_k + \tau \mathbf{M}^{-1}\mathbf{P}_{k+\frac{1}{2}}$

2 - calculate $\mathbf{F}_{int\,k+1}$ with $\mathbf{F}_{int\,k+1}(\mathbf{U}_{k+1})$

3 - calculate $\dot{\mathbf{g}}_{N_{free}}$ and $\mathbf{P}_{free}$ with $\dot{\mathbf{g}}_{N_{free}} = -\mathbf{L}\mathbf{M}^{-1}\mathbf{P}_{free}$ and $\mathbf{P}_{free} = \mathbf{P}_{k+\frac{1}{2}} + \tau(\mathbf{F}_{ext\,k+1} + \mathbf{F}^{s}_{ext\,k+1} - \mathbf{F}_{int\,k+1}) - \mathbf{L}^T\mathbf{\Lambda}_{k+\frac{1}{2}}$

4 - calculate $\mathbf{\Lambda}_{k+\frac{3}{2}}$ with $\Lambda^i_{k+\frac{3}{2}} = \begin{cases} 0 \quad \text{if} \quad g^i_{N\,k+1} > 0 \\ max(0, H^{-1}_{ii}\dot{g}_{N}{}^{i}_{free}) \end{cases}$

5 - calculate $\mathbf{P}_{k+\frac{3}{2}}$ with $\mathbf{P}_{k+\frac{3}{2}} = \mathbf{P}_{free} + \mathbf{L}^T(\mathbf{\Lambda}_{k+\frac{3}{2}})$

6 - calculate $\dot{\mathbf{g}}_{N_{k+\frac{3}{2}}}$ with $-\mathbf{L}\mathbf{M}^{-1}\mathbf{P}_{k+\frac{3}{2}}$ (only for post-processing energy balance)

Variational formulation discretized in space and time for non-smooth elastodynamics

- *Multi-symplectic form discretized in space and time for non-smooth elastodynamics:*

$$\boxed{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\tau}(\mathbf{U}_{k+1} - \mathbf{U}_k) \\ \frac{1}{\tau}(\mathbf{P}_{k+\frac{3}{2}} - \mathbf{P}_{k+\frac{1}{2}}) \\ d_t \mathbf{F}_{int} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{k+1} \\ d_{\mathbf{x}} \mathbf{p} \\ \mathbf{F}_{int\,k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{ext\,k+1} + \mathbf{F}^s_{ext\,k+1} \\ \mathbf{M}^{-1}\mathbf{P}_{k+\frac{1}{2}} \\ -\mathbf{K}^{-1}_{tang}\mathbf{F}_{int\,k+1} \end{bmatrix} + \begin{bmatrix} \frac{1}{\tau}\mathbf{L}^T(\mathbf{\Lambda}_{k+\frac{3}{2}} - \mathbf{\Lambda}_{k+\frac{1}{2}}) \\ 0 \\ 0 \end{bmatrix}}$$

The proposed explicit variational time integrator preserves the multi-symplectic discretized form of non-regular elastodynamics. As a result, ***the linear and angular momentum equations are verified exactly****, even in the presence of Lagrange multipliers. Nevertheless, it can be shown that* ***the discretized energy balance is not guaranteed exactly****. However, it can be shown that the corresponding error decays exponentially, which characterizes the high robustness of variational time integrators over long times. To ensure an exact discretized energy balance, a non-uniform time step must be considered.*

Variational formulation discretized in space and time for non-smooth elastodynamics

- *Multi-field action discretized in space and time for non-smooth elastodynamics with adaptive time step:*

$$\tau_{k+1} = t_{k+1} - t_k$$

- *Multi-field action discretized in space and time for non-smooth elastodynamics:*

$$\boxed{\begin{aligned}
&\overset{\sim}{\mathcal{A}}^{h}(t) \simeq \sum_{k=0}^{N-1} \overset{\sim}{\mathcal{A}}^{h\tau}(\tau_{k+1}, \mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k+1}, \mathbf{\Lambda}_{k+\frac{1}{2}})\\
&\overset{\sim}{\mathcal{A}}^{h\tau}(\tau_{k+1}, \mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k+1}, \mathbf{\Lambda}_{k+\frac{1}{2}}))\\
&= \tfrac{1}{2}(\tau_k + \tau_{k+1})\mathbf{P}^T_{k+\frac{1}{2}}.\frac{(\mathbf{U}_{k+1} - \mathbf{U}_k)}{\tau_{k+1}} - \tfrac{1}{2}(\tau_k \mathbf{F}_{int\,k}^{\;T}.\mathbf{U}_k + \tau_{k+1}\mathbf{F}_{int\,k+1}^{\;T}.\mathbf{U}_{k+1})\\
&-\tfrac{1}{2}(\tau_k \mathcal{L}^{h\tau *}(\mathbf{U}_k, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k}) + \tau_{k+1}\mathcal{L}^{h\tau *}(\mathbf{U}_{k+1}, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k+1}))\\
&+\tfrac{1}{2}(\tau_k + \tau_{k+1})\mathbf{\Lambda}^T_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}}\\
&+ \text{HSM conditions}
\end{aligned}}$$

with: $\mathcal{L}^{h\tau *}(\mathbf{U}_k, \mathbf{P}_{k+\frac{1}{2}}, \mathbf{F}_{int\,k}) = T^{h\tau *}(\mathbf{P}_{k+\frac{1}{2}}) - \Psi^{h\tau *}(\mathbf{F}_{int\,k}) - \mathbf{F}^T_{ext\,k}.\mathbf{U}_k$

Variational formulation discretized in space and time for non-smooth elastodynamics

- State vector and corresponding Euler–Lagrange equations:

$$(\tau_{k+2}, \dot{\mathbf{g}}_{N_{k+\frac{3}{2}}} \mathbf{U}_{k+2}, \mathbf{P}_{k+\frac{3}{2}}, \mathbf{F}_{int\,k+2}, \boldsymbol{\Lambda}_{k+\frac{3}{2}})$$

$$\boxed{\begin{aligned}
&\mathbf{U}_{k+1} = \mathbf{U}_k + \tau_{k+1}\mathbf{M}^{-1}\mathbf{P}_{k+\frac{1}{2}} \\
&\mathbf{U}_{k+1} - \mathbf{K}_{tang}^{-1}\mathbf{F}_{int\,k+1} = 0 \\
&\frac{1}{2}\frac{\tau_{k+1}+\tau_{k+2}}{\tau_{k+2}}\mathbf{P}_{k+\frac{3}{2}} = \frac{1}{2}\frac{\tau_k+\tau_{k+1}}{\tau_{k+1}}\mathbf{P}_{k+\frac{1}{2}} + \tau_{k+1}(\mathbf{F}_{ext\,k+1} + \mathbf{F}^s_{ext\,k+1} - \mathbf{F}_{int\,k+1}) \\
&+\frac{1}{2}\frac{\tau_{k+1}+\tau_{k+2}}{\tau_{k+2}}\boldsymbol{\Lambda}_{k+\frac{3}{2}} - \frac{1}{2}\frac{\tau_k+\tau_{k+1}}{\tau_{k+1}}\mathbf{L}^T\boldsymbol{\Lambda}_{k+\frac{1}{2}} \\
&+ \text{HSM conditions} \\
&\tau_{k+2} = (\tfrac{1}{2}\mathbf{P}^T_{k+\frac{3}{2}}.(\mathbf{U}_{k+2} - \mathbf{U}_{k+1})) \\
&.(-\tfrac{1}{2}\tfrac{-\tau_k}{\tau^2_{k+1}}\mathbf{P}^T_{k+\frac{1}{2}}.(\mathbf{U}_{k+1} - \mathbf{U}_k) + \mathbf{F}_{int\,k+1}^T.\mathbf{U}_{k+1} \\
&+\tfrac{1}{2}(T^{h\tau^*}(\mathbf{P}_{k+\frac{3}{2}}) + T^{h\tau^*}(\mathbf{P}_{k+\frac{1}{2}})) - \Psi^{h\tau^*}(\mathbf{F}_{int\,k+1}) \\
&-\mathbf{F}^T_{ext\,k+1}.\mathbf{U}_{k+1} + \tfrac{1}{2}(\boldsymbol{\Lambda}^T_{k+\frac{3}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{3}{2}}} + \boldsymbol{\Lambda}^T_{k+\frac{1}{2}}.\dot{\mathbf{g}}_{N_{k+\frac{1}{2}}}))^{-1}
\end{aligned}}$$

Variational formulation discretized in space and time for non-smooth elastodynamics

- *Explicit variational integrator with adaptive time step:*

---

0 - time step predictor $\tau_{k+2} = \tau_{k+1}$

1 - calculate $\dot{\mathbf{g}}_{N_{free}}$ and $\mathbf{P}_{free}$ with $\dot{\mathbf{g}}_{N_{free}} = -\mathbf{L}\mathbf{M}^{-1}\mathbf{P}_{free}$ and $\frac{1}{2}\frac{\tau_{k+1}+\tau_{k+2}}{\tau_{k+2}}\mathbf{P}_{free} = \frac{1}{2}\frac{\tau_k+\tau_{k+1}}{\tau_{k+1}}\mathbf{P}_{k+\frac{1}{2}} + \tau_{k+1}(\mathbf{F}_{ext\,k+1} + \mathbf{F}^s_{ext\,k+1} - \mathbf{F}_{int\,k+1}) - \frac{1}{2}\frac{\tau_k+\tau_{k+1}}{\tau_{k+1}}\mathbf{L}^T\boldsymbol{\Lambda}_{k+\frac{1}{2}}$

2 - calculate $\boldsymbol{\Lambda}_{k+\frac{3}{2}}$ with $\Lambda^i_{k+\frac{3}{2}} = \begin{cases} 0 \quad \text{if} \quad g^i_{N\,k+1} > 0 \\ max(0, H^{-1}_{ii}\dot{g_N}^i_{free}) \end{cases}$

3 - calculate $\mathbf{P}_{k+\frac{3}{2}}$ with $\mathbf{P}_{k+\frac{3}{2}} = \mathbf{P}_{free} + \mathbf{L}^T(\boldsymbol{\Lambda}_{k+\frac{3}{2}})$

4 - calculate $\dot{\mathbf{g}}_{N_{k+\frac{3}{2}}}$ with $-\mathbf{L}\mathbf{M}^{-1}\mathbf{P}_{k+\frac{3}{2}}$ (for time step update)

5 - calculate $\mathbf{U}_{k+2}$ with $\mathbf{U}_{k+2} = \mathbf{U}_{k+1} + \tau_{k+2}\mathbf{M}^{-1}\mathbf{P}_{k+\frac{3}{2}}$

6 - calculate $\mathbf{F}_{int\,k+2}$ with $\mathbf{F}_{int\,k+2}(\mathbf{U}_{k+2})$

7 - update time step τ_{k+2} with equation (116) and go to step 1 if $\tau_{k+2} \neq \tau_{k+1}$

Variational formulation discretized in space and time for non-smooth elastodynamics

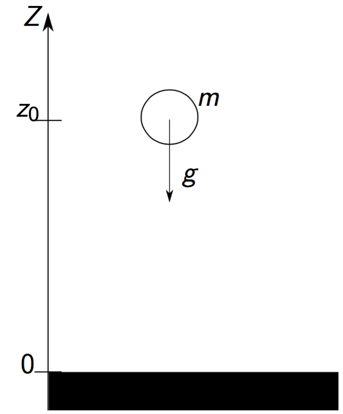
- *Multi-symplectic form discretized in space and time for non-smooth elastodynamics with adaptive time step:*

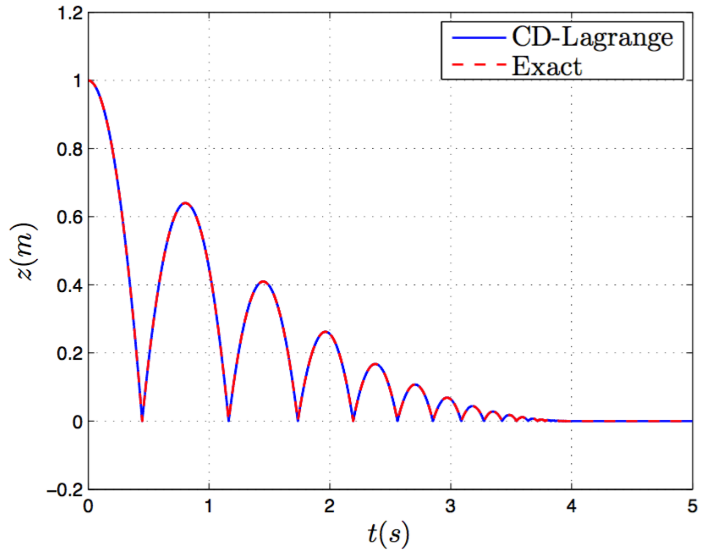
$$
\boxed{
\begin{aligned}
&\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \frac{1}{\tau_{k+2}}(\mathbf{U}_{k+2} - \mathbf{U}_{k+1}) \\ \frac{1}{2}\frac{\tau_{k+1}+\tau_{k+2}}{\tau_{k+2}}\mathbf{P}_{k+\frac{3}{2}} - \frac{1}{2}\frac{\tau_{k}+\tau_{k+1}}{\tau_{k+1}}\mathbf{P}_{k+\frac{1}{2}} \\ d_t\mathbf{F}_{int} \end{bmatrix}
+ \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} \mathbf{U}_{k+2} \\ d_{\mathbf{x}}\mathbf{p} \\ \mathbf{F}_{int\,k+2} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{F}_{ext\,k+2} + \mathbf{F}^{s}_{ext\,k+2} \\ \mathbf{M}^{-1}\mathbf{P}_{k+\frac{3}{2}} \\ -\mathbf{K}^{-1}_{tang}\mathbf{F}_{int\,k+2} \end{bmatrix}
+ \begin{bmatrix} \frac{1}{2}\frac{\tau_{k+1}+\tau_{k+2}}{\tau_{k+2}}\mathbf{L}^{T}\mathbf{\Lambda}_{k+\frac{3}{2}} - \frac{1}{2}\frac{\tau_{k}+\tau_{k+1}}{\tau_{k+1}}\mathbf{L}^{T}\mathbf{\Lambda}_{k+\frac{1}{2}} \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
}
$$

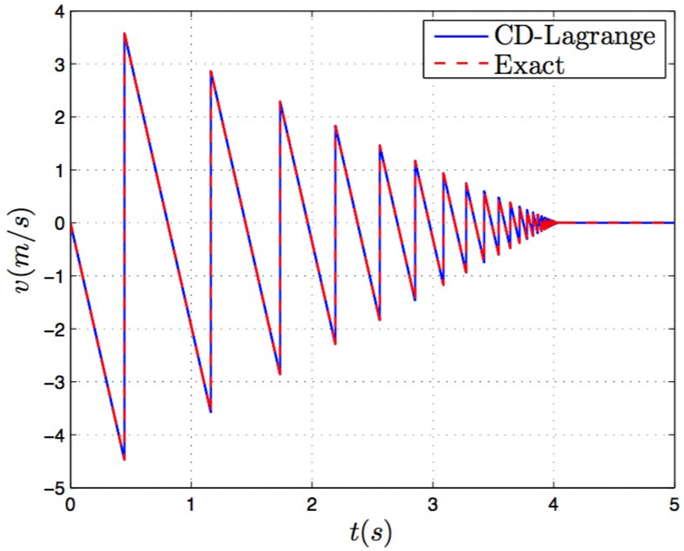
The proposed explicit variational time integrator checks the following properties:

- Explicit integrator of the "time stepping" type
- Non-smooth space-time convergence in the Hausdorff sense
- multi-symplectic discretized form (conservation of linear and angular momentum)
- Discrete energy conservation
- automatic adaptive time step (with CFL condition checking)
- Matrix free (explicit LCP) for deformable/rigid type contacts
- Great robustness over long times with a very large number of impacts

- ***The bouncing ball example: rigid / rigid analytical solution***
  - ***First rebound:*** $t_0 = \sqrt{\frac{2h}{g}}$
  - ***Nth rebound:*** $t_n = t_0\left(2\frac{1-e^n}{1-e} - 1\right)$
  - ***Downtime of the mass:*** $T = t_0 \frac{1+e}{1-e}$

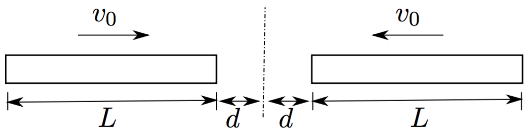




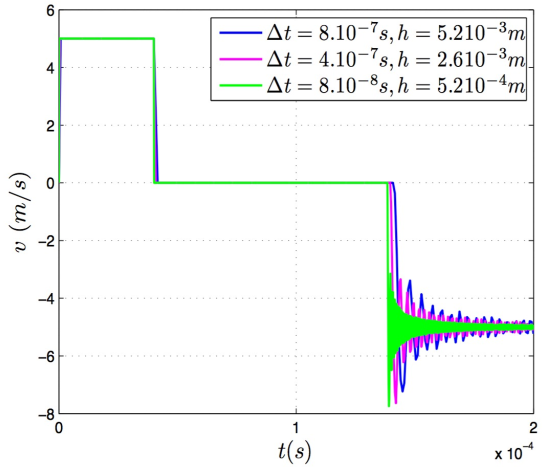


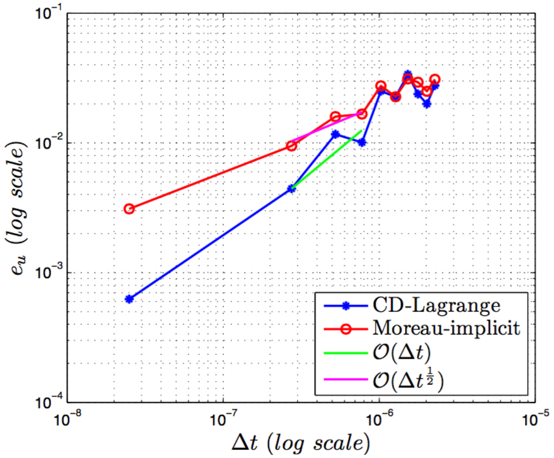
The mass does an infinity of rebounds before it stops in a finite time (Zeno Phenomenon) (restitution coefficient e=0.8)

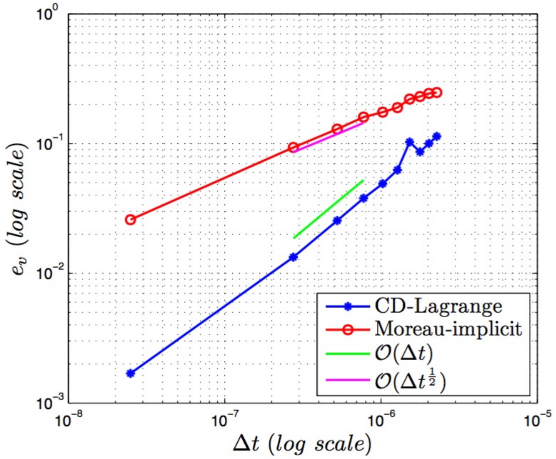
- ***Contact / impact of two identical elastic bars : deformable / deformable analytical solution***



$E = 2.1\ 10^{11}\ Pa$
$\rho = 7847\ kg/m^3$
$L = 0.254\ m$
$d = 0.2\ 10^{-3}\ m$
$v_0 = 5\ m/s$
$A = 0.645\ 10^{-3}\ m^2$





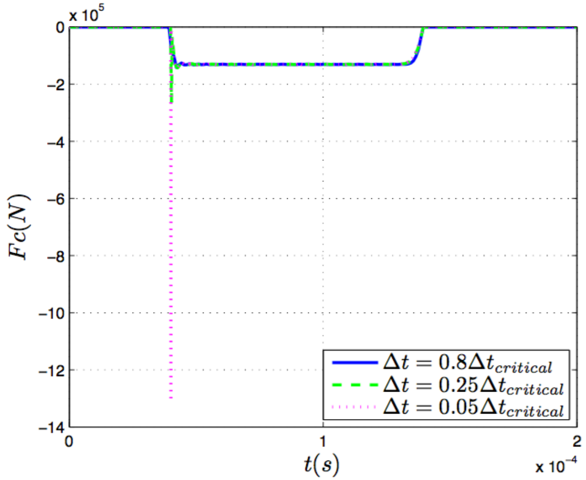


→ ***Space-time global error indicator (Hausdorff measure)*** ***[Acary 2012]***

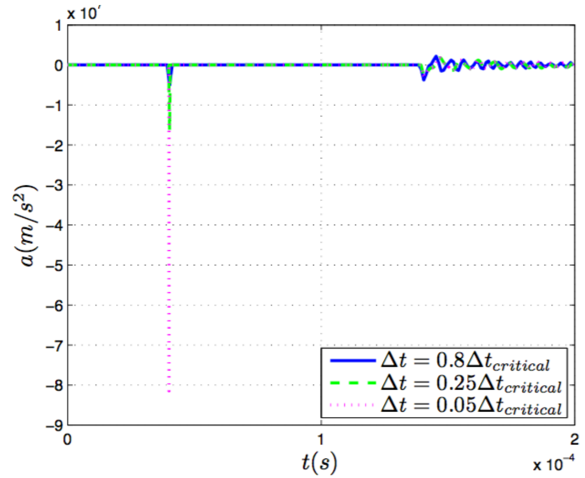
→ ***Moreau implicit of order ½, CD-Lagrange of order 1*** ***[Fekak 2017]***

- **Contact / impact of two identical elastic bars : deformable / deformable analytical solution**

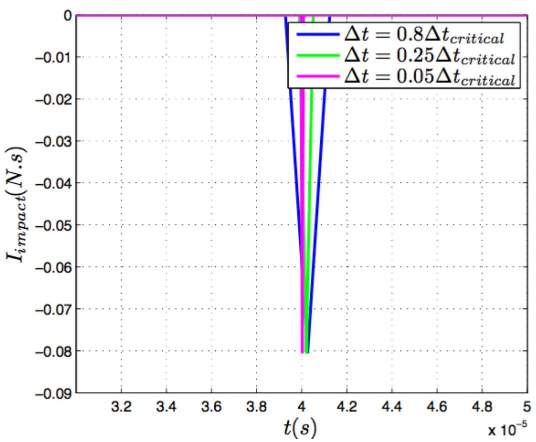
Contact force



Contact acceleration



Impact impulse



At impact time, acceleration and contact forces do not converge, however impulse converges.

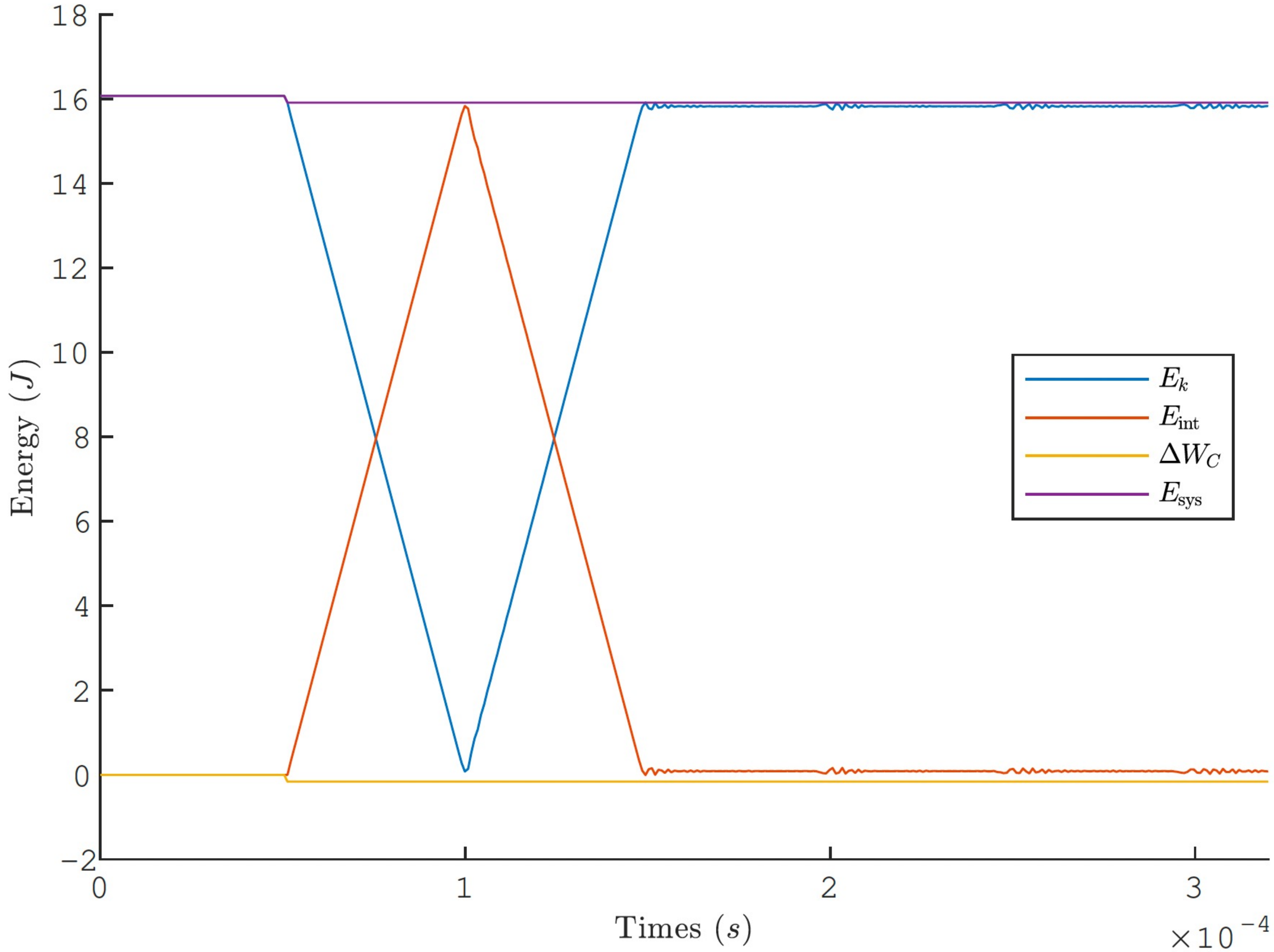


Figure: The impacting bar: energy balance of CD-Lagrange

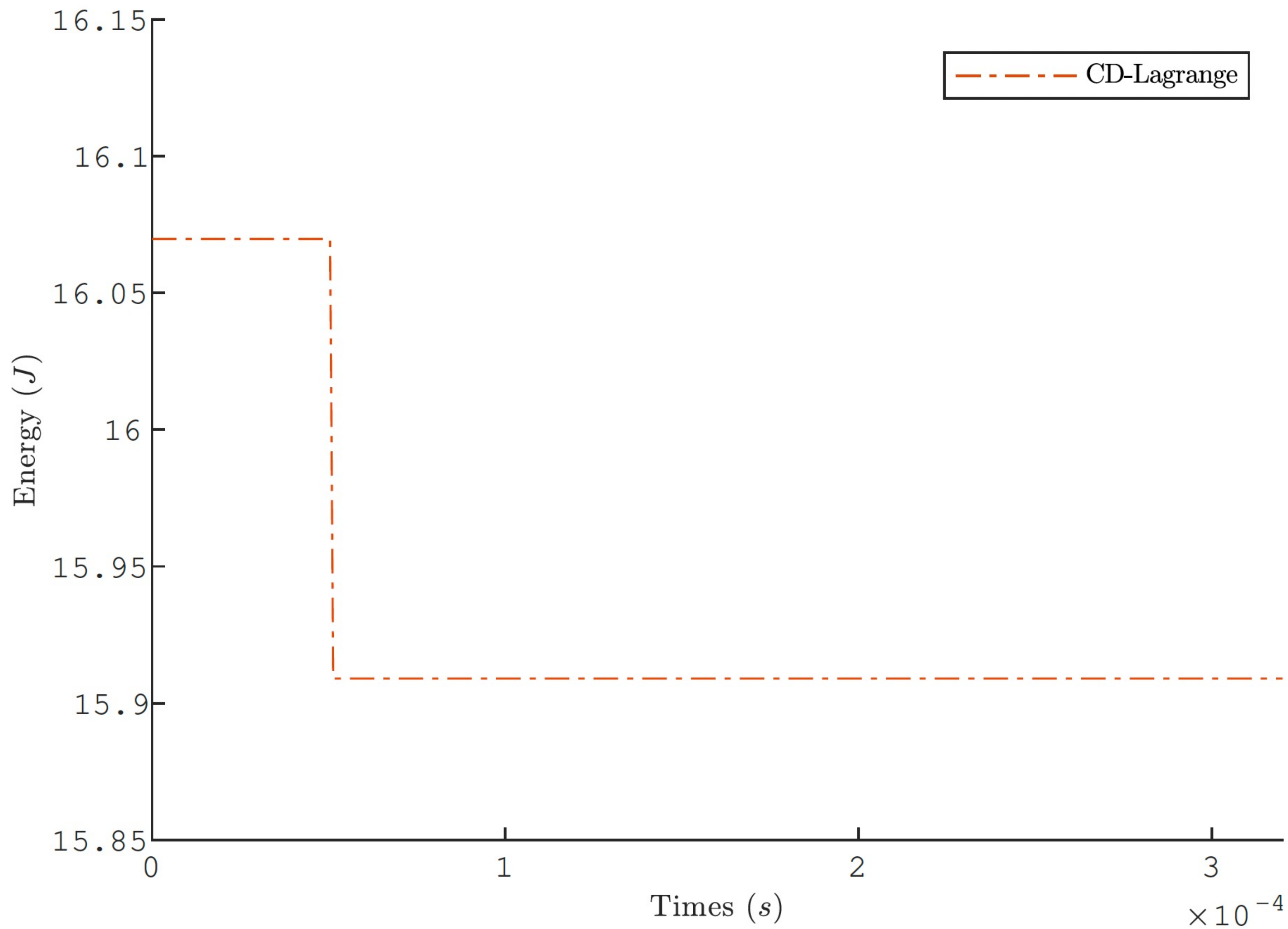


Figure: The impacting bar: system energy of CD-Lagrange

- *Discretized energy balance with singular mass / adaptive time step*

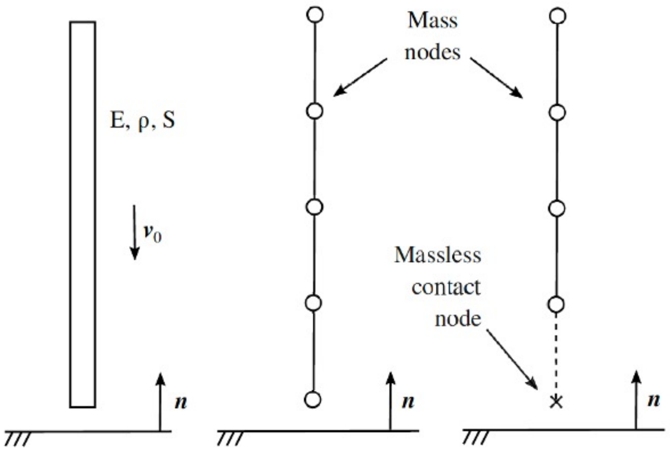


Figure: Consistent and singular spatial discretization

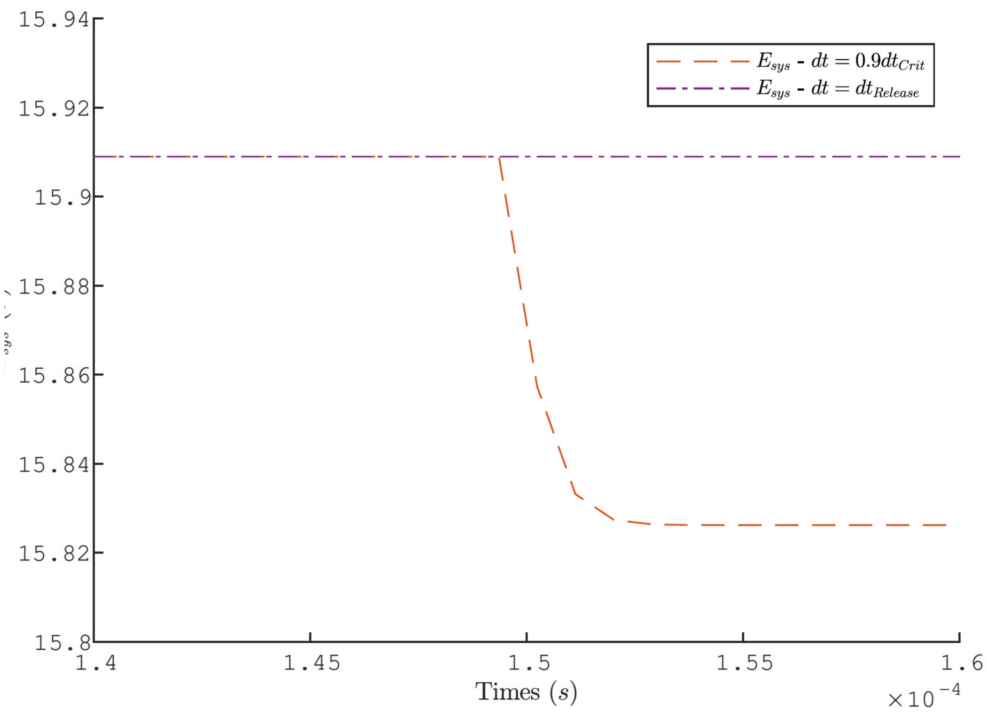


Figure: Contact node - $\Delta E_k = \Delta \bar{W}_{\text{int}} + \Delta \widetilde{W}_{\text{int}}$ for different t_{Release}

The rotating spring: a focus on discrete angular momentum

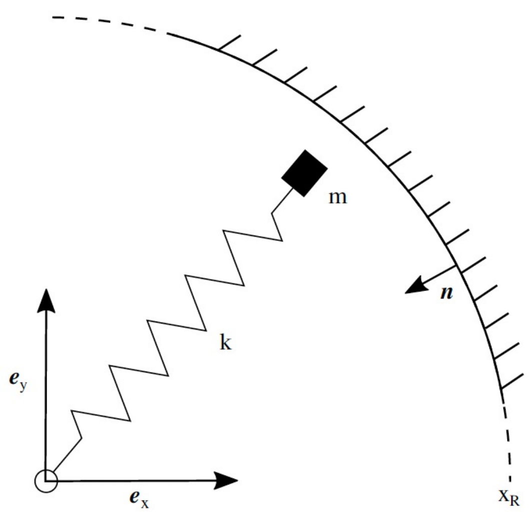


Figure: The rotating spring

$$m\mathrm{d}\dot{\boldsymbol{x}} + k\left(1 - \frac{l_0}{\|\boldsymbol{x}\|}\right)\boldsymbol{x} = \mathrm{d}r\boldsymbol{n} \qquad (12)$$

A mass-spring system in rotation:

- impact with a restitution coefficient e_c;
- large rotations;

What are the highlighted properties ?

- conservation of discrete momentum in smooth phase;
- conservation of discrete momentum through non-smooth events.

A case which challenges the symplecticity under non-smooth events.

The Moreau-Jean scheme[2] as a reference (time-integration with $\theta = 1/2$).

[2] Jean, M. The non-smooth contact dynamics method. *Computer Methods in Applied Mechanics and Engineering* **177**, 235–257 (1999).

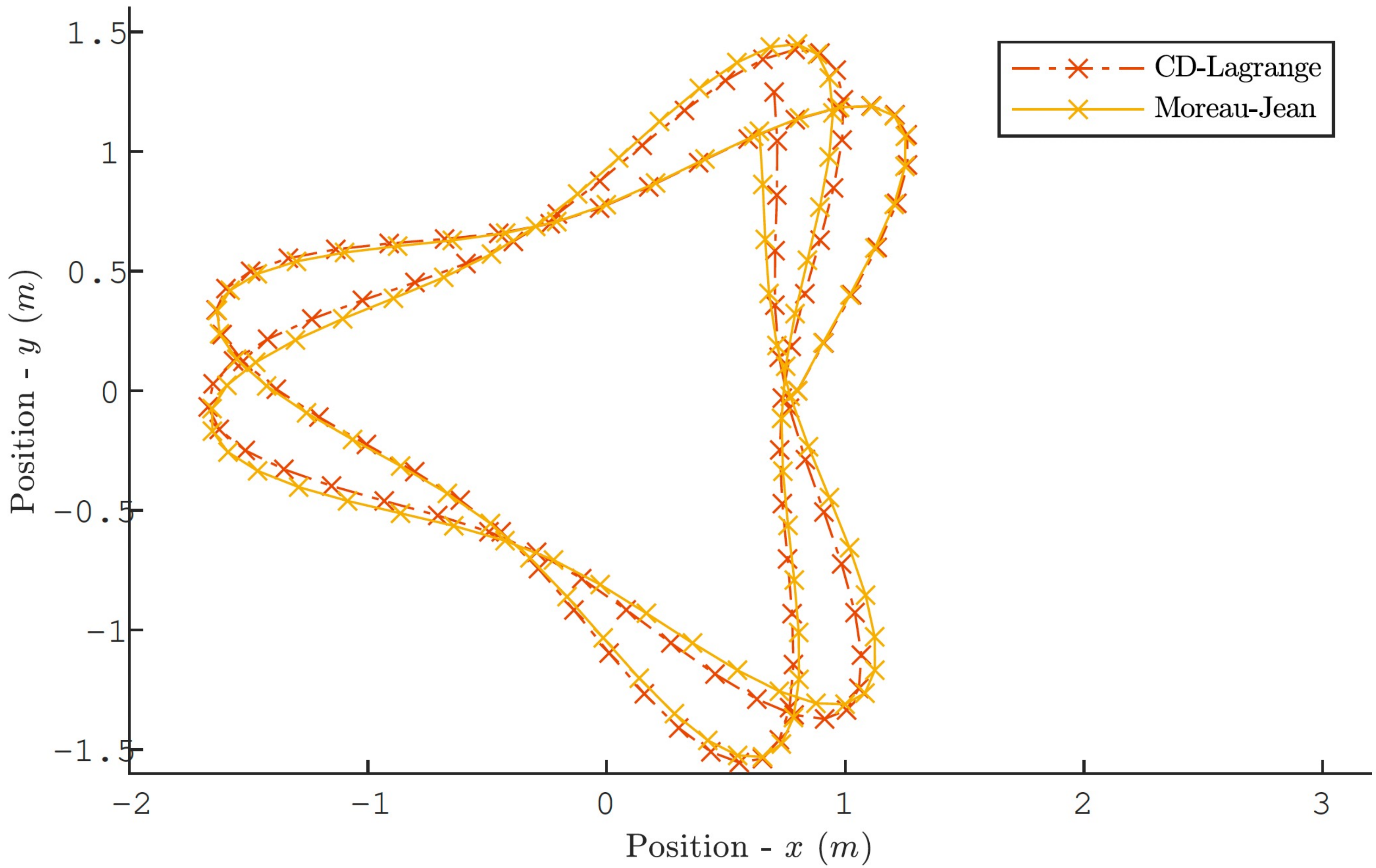


Figure: The rotating spring: position (over 10 s) - $e_c = 1$

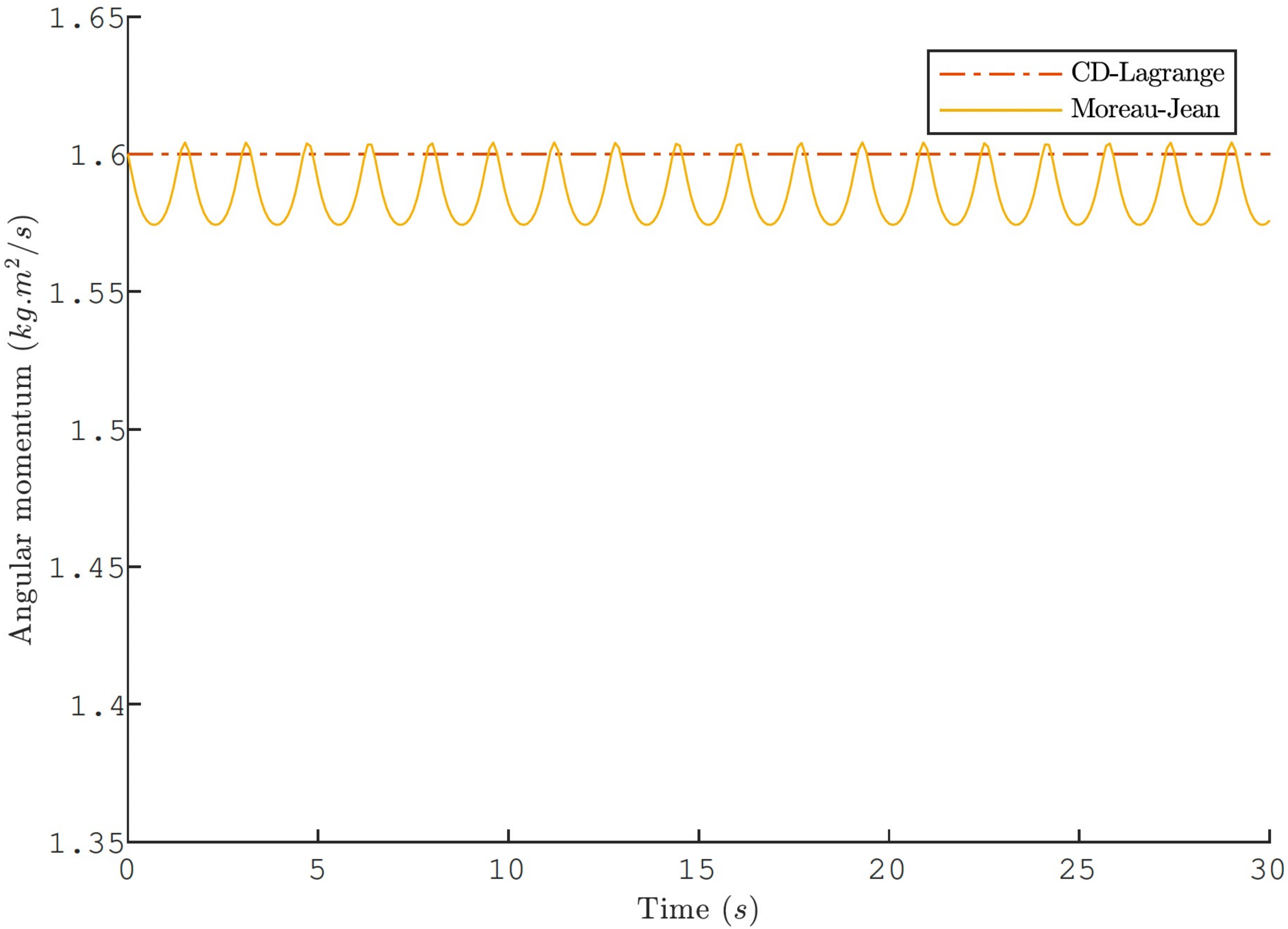


Figure: The rotating spring: angular momentum - $e_c = 1$

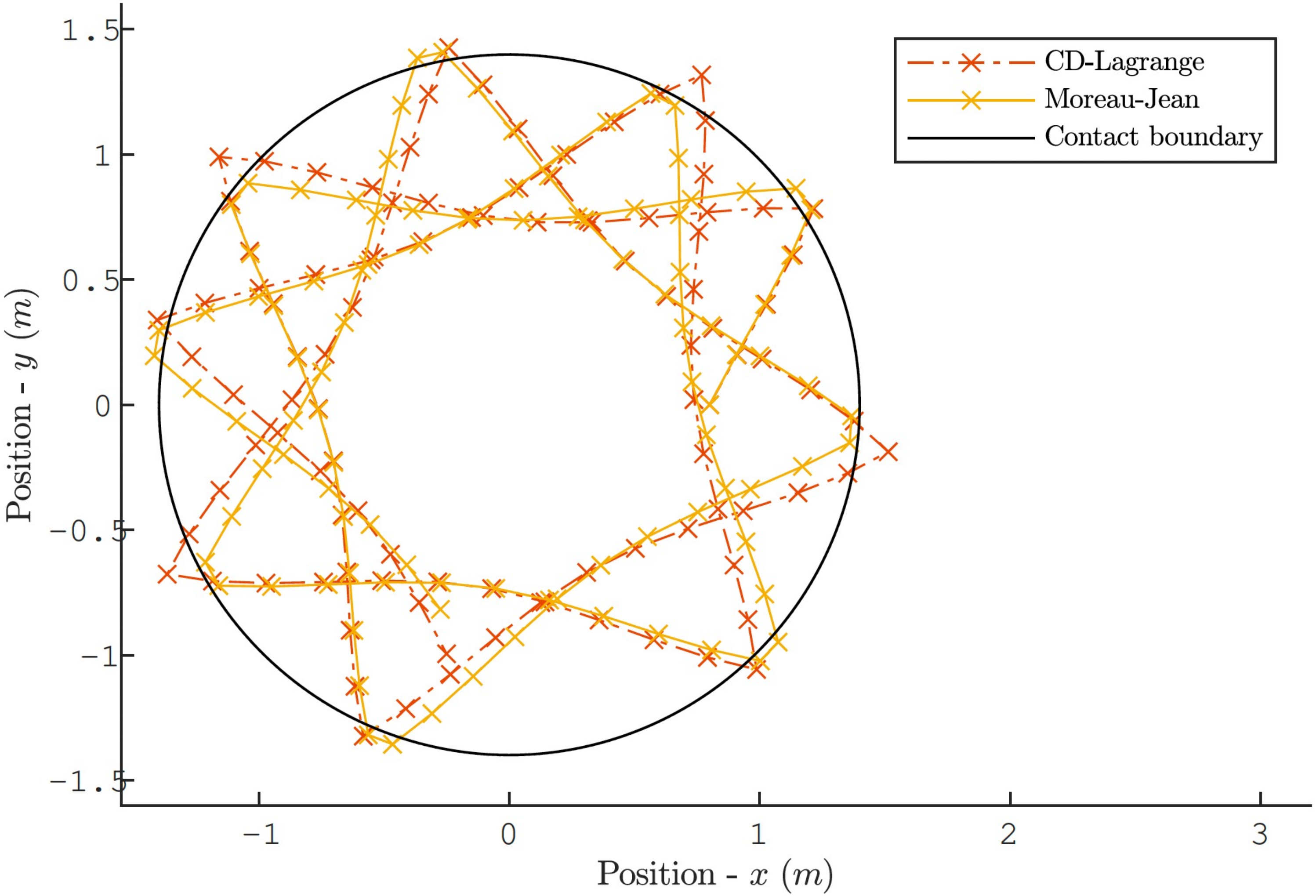


Figure: The rotating spring: position (over 10 s) - $e_c = 1$

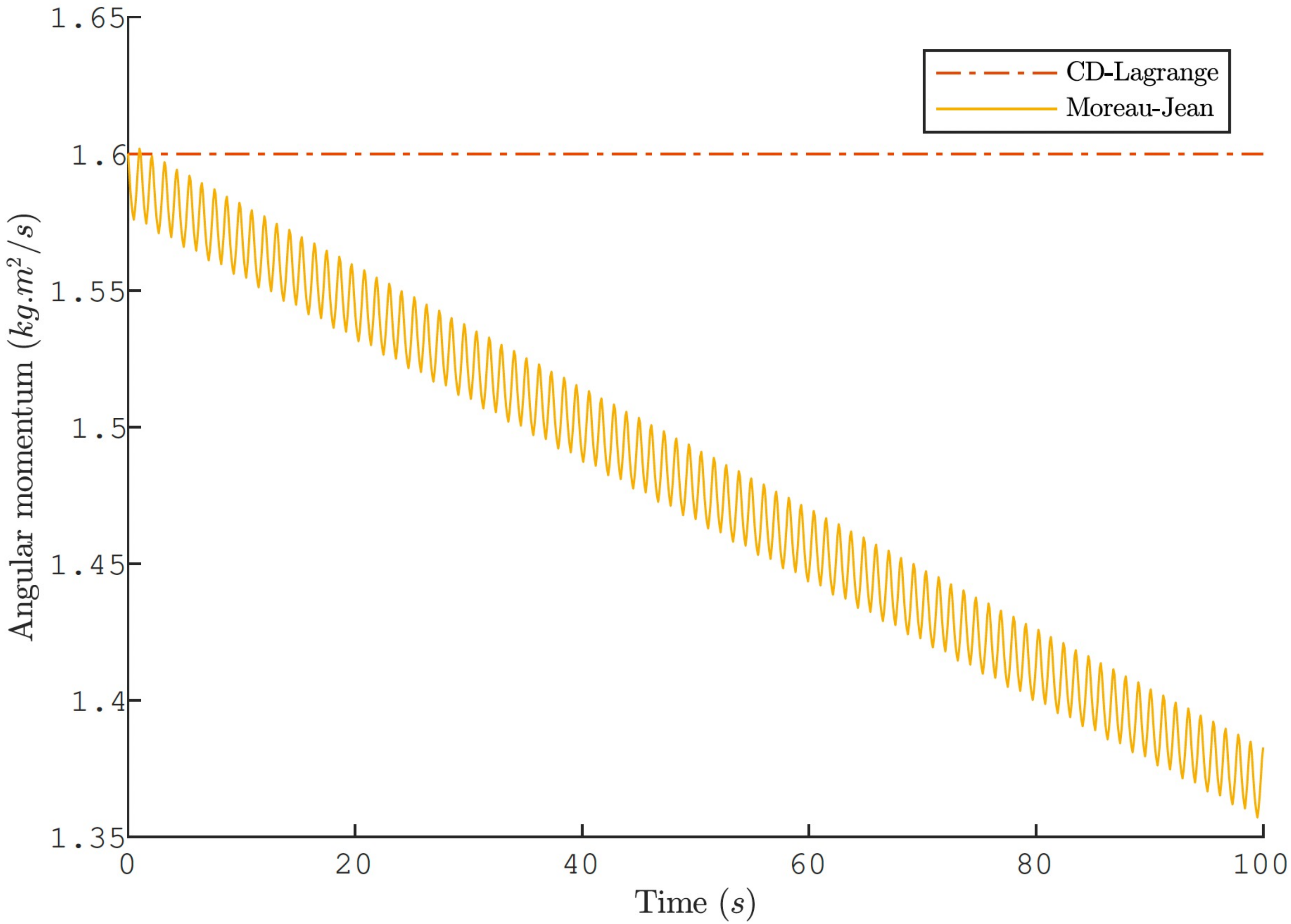
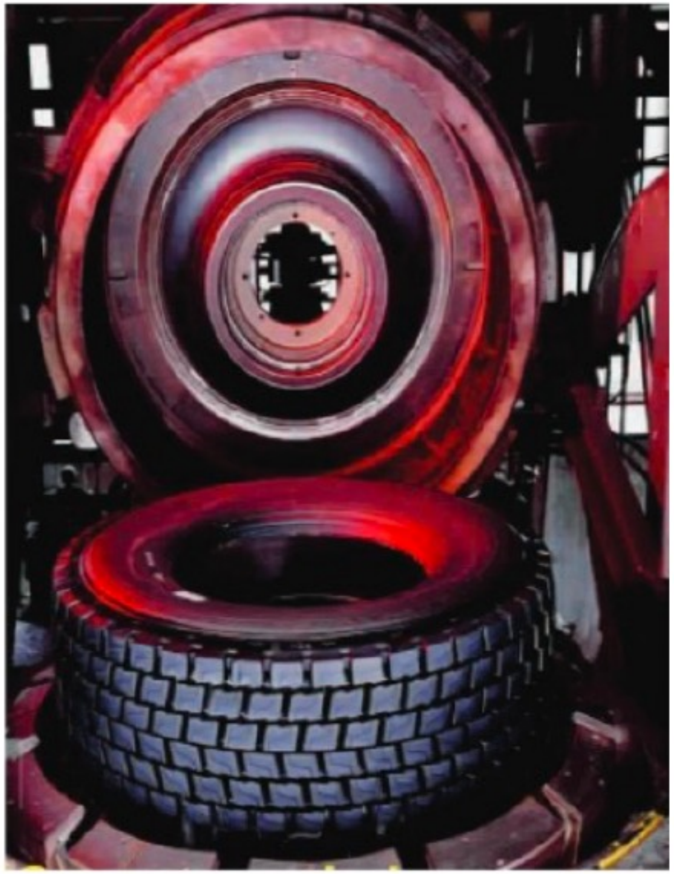


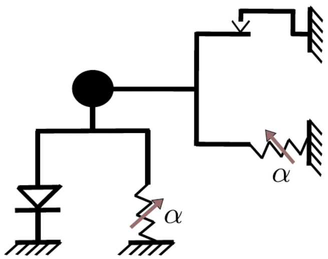
Figure: The rotating spring: angular momentum - $e_c = 1$

The unmolding tyre process

P. Larousse, J. Di Stasio, D. Dureisseix, A. Gravouil



RCCM model: interface model with contact, friction, adhesion and damage



3D rheology model of the RCCM model

- Normal contact behaviour

$$\begin{cases} \textbf{if} & g_{n+1} > 0\textbf{, then } r_{c,N,n+3/2} = 0 \\ \textbf{else} & g_{n+1} = 0\textbf{, and } 0 \leq r_{c,N,n+3/2} \perp v_{c,N,n+3/2} \geq 0 \end{cases}$$

- Tangential contact behaviour

$$\begin{cases} \textbf{if} & g_{n+1} > 0\textbf{, then } r_{c,T,n+3/2} = 0 \\ \textbf{else} & g_{n+1} = 0\textbf{, and} \\ 0 \leq (\mu r_{c,N,n+3/2} - & r_{c,T,n+3/2}) \perp \|\mathbf{v}_{c,T,n+3/2}\| \geq 0 \end{cases}$$

- Damaging elastic interface behaviour

$$\begin{cases} \textbf{if } [u]_{n+1} \geq \alpha_{n+1} \textbf{ and } \alpha_{n+1} & < u_r, \textbf{ then} \\ & \dot{\alpha}_{n+\frac{3}{2}} = < \dot{f}_{n+\frac{3}{2}} >_+ \\ \textbf{else} \quad \dot{\alpha}_{n+\frac{3}{2}} = 0 & \end{cases}$$

$$f([\mathbf{u}]) = \sqrt{[\mathbf{u}]_N + \lambda [\mathbf{u}]_T}$$

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RCCM model: interface model with contact, friction, adhesion and damage

- New mechanical equation including damage

$$\begin{aligned}\mathbf{M}(\mathbf{V}_{n+3/2}-\mathbf{V}_{n+1/2}) = \; & h(\mathbf{F}_{\mathrm{ext},n+1}-\mathbf{F}_{\mathrm{int},n+1}) \\ & +\mathbf{L}_N^T\mathbf{R}_{c,N,n+3/2}+\mathbf{L}_T^T\mathbf{R}_{c,T,n+3/2} \\ & +h\;\mathbf{F}_{d,n+3/2}\end{aligned}$$

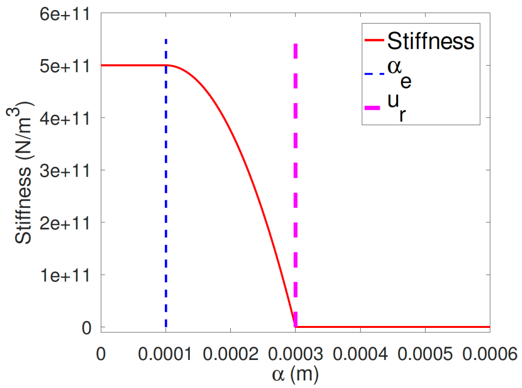
  - Damage associated force

$$\mathbf{F}_{d,n+3/2} = \mathbf{S}\;(\mathbf{L}_N^T\mathbf{R}_{d,N,n+1}+\mathbf{L}_T^T\mathbf{R}_{d,T,n+1})$$

  - Elastic damage distribution

$$\begin{cases}\mathbf{R}_{d,N,n+1} & = g_N(\alpha_{n+1})\mathbf{L}_N[\mathbf{u}]_{n+1} \\ \mathbf{R}_{d,T,n+1} & = g_T(\alpha_{n+1})\mathbf{L}_T[\mathbf{u}]_{n+1}\end{cases}$$

- **S** is a diagonal matrix of nodal weights



A new RCCM model: interface model with contact, friction, adhesion and damage with delay effect

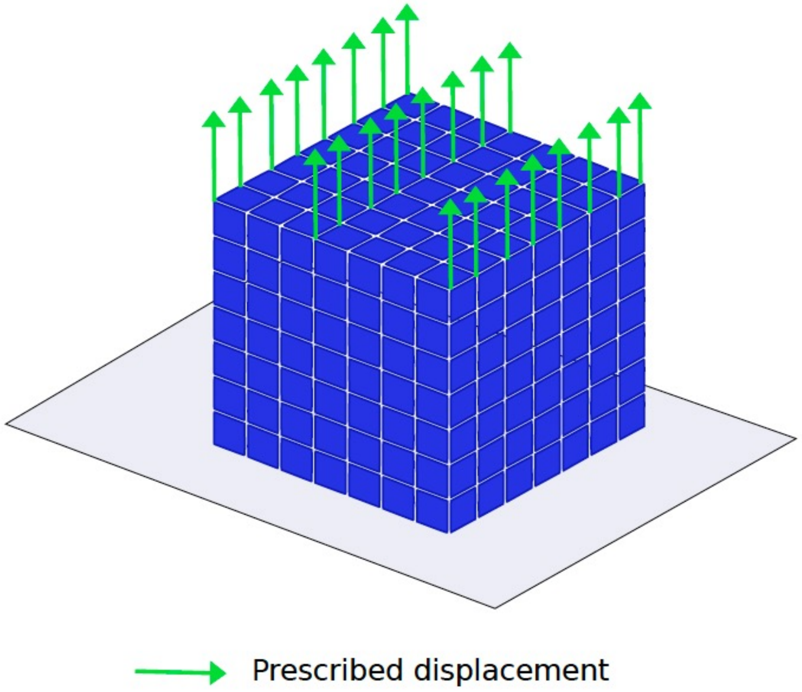
without delay effect

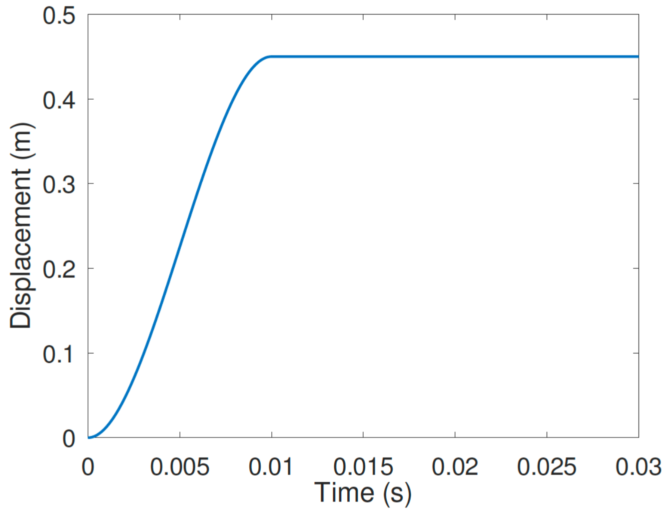
$$\begin{cases} \textbf{if } f([\mathbf{u}]) = \alpha \textbf{ and } \alpha < u_r, \quad \dot{\alpha} =< \dot{f} >_+ \\ \textbf{else} \quad \dot{\alpha} = 0 \end{cases}$$

with delay effect

$$\begin{cases} \textbf{if } f([\mathbf{u}]) = \alpha \textbf{ and } \alpha < u_r, \\ \quad \dot{\alpha} = v_r \left[1 - \exp\left(-\frac{1}{v_r} < \dot{f} >_+\right)\right] \\ \textbf{else} \quad \dot{\alpha} = 0 \end{cases}$$

A dynamic tensile test on a cube with RCCM interface

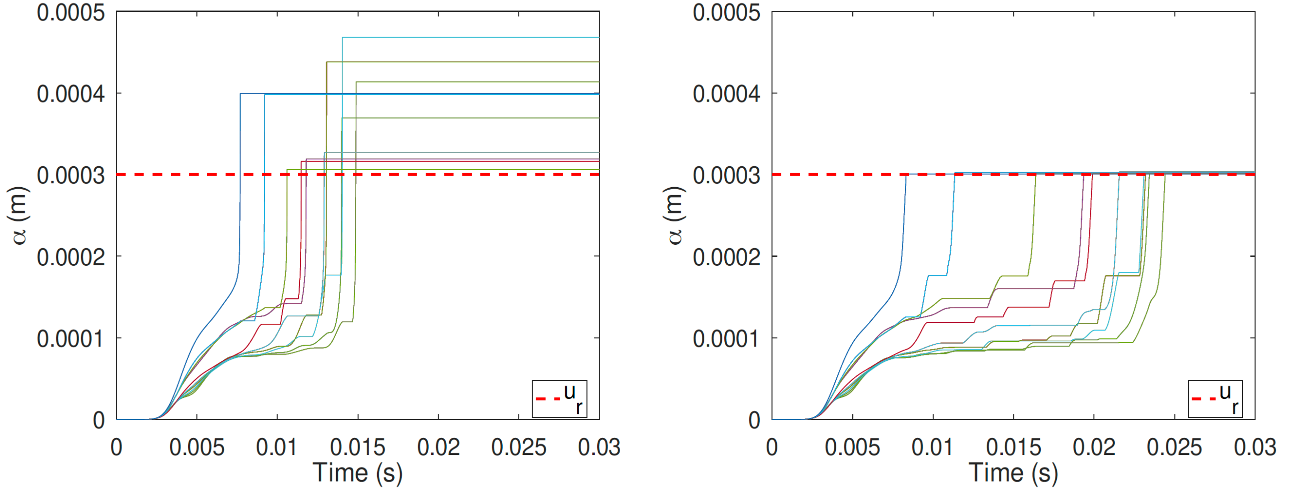




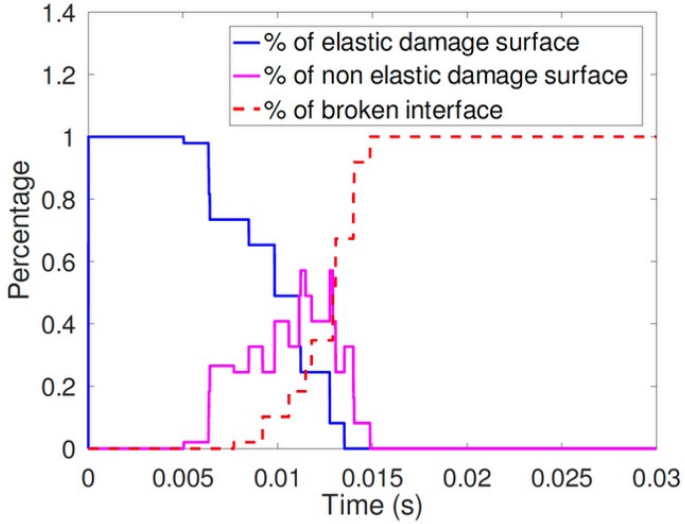
Prescribed displacement evolution on the cube top face.

A dynamic tensile test on a cube with RCCM interface

- Comparison of the damage evolution



Damage variable evolution for each node without delayed effect (left) and with delayed effect (right).



Percentage of surface states.

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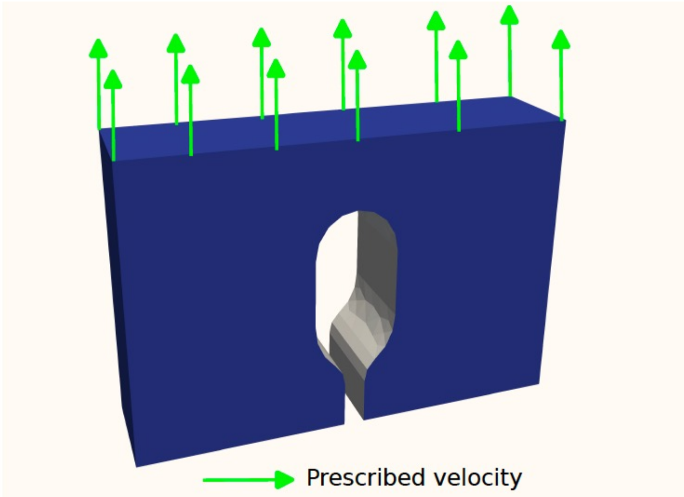
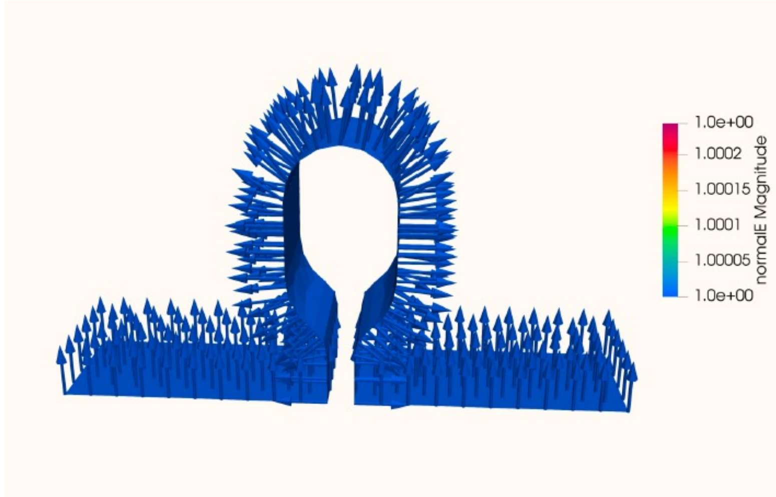
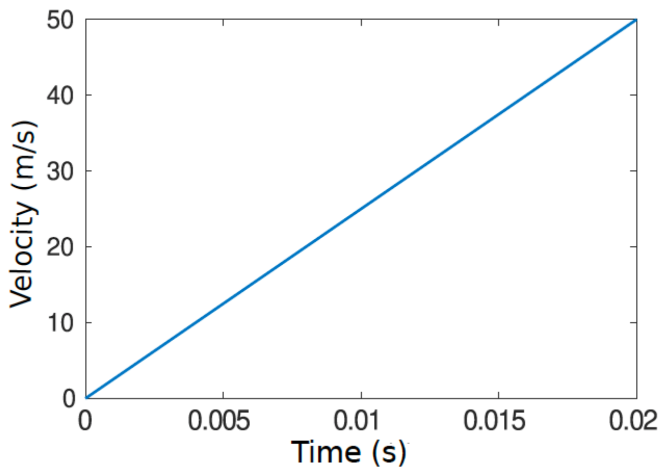
Industrial application: the "water drop" mold shape







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